Applied Business

Issues & Solutions

Applied Business: Issues & Solutions, 2(2023)
ISSN 2783-6967
https://doi.org/10.57005/ab.2023.2

CONTENT

Article

Foreign Direct Investment's Impact on the Activity of Transnational Corporations

Olga Nosova[1,2*]
[1]Department of Business, Vilnius Business College, Kalvarijų g. 129-401, LT-08221 Vilnius, Lithuania
[2]Faculty Public Governance and Business, Mykolas Romeris University,
Ateities g. 20, LT-08303, Vilnius, Lithuania
[*] Corresponding author, e-mail: *olgano59@gmail.com*

Received: 20 October 2023
Accepted: 10 November 2023
Online: 30 December 2023

Abstract. The purpose of this article is to analyse how FDI impacts transnational corporation, local firms' productivity and investment climate, and its influence on domestic enterprises. Positive and negative effects of foreign capital movement on the economy of the recipient country are examined. The evaluation of FDI flows, variations in global FDI by country, and net annual FDI flows are considered while determining global capital movement policies. Direct and indirect FDI spillover effects on domestic firms across countries are considered factors of production. The challenges faced by governments in creating policies to attract FDI practices in emerging economies are reflected. An attractive policy that encourages the expansion of foreign capital is proposed.

JEL: F20; F21; F23.

Keywords: foreign direct investment; investment climate; global capital movement; foreign capital; transnational corporation.

Citation: Olga Nosova (2023) Foreign Direct Investment's Impact on the Activity of Transnational Corporations. – *Applied Business: Issues & Solutions* 2(2023)3–9 – ISSN 2783-6967.
https://doi.org/10.57005/ab.2023.2.1

Introduction

The study of foreign direct investment (FDI) effects on the productivity of local firms addresses a research effort to assess the impacts of enhanced FDI activity within emerging economies. To design a more effective policy to promote alluring FDI practices, it is important to study the consequences of FDI on the productivity of local businesses. The increase of foreign capital inflow in recent years demonstrates that workers in industries with greater foreign participation experience faster wage growth. The actual relationship between horizontal and vertical FDI spillover effects remains unclear, although the available research has identified some positive correlations.

Recent studies have highlighted the considerable research efforts in developing understanding about the investment motivation of FDI spillover effects, its impact on economic growth, competitive advantages inside the developed economies and draw attention to short-term adjustment problems rather than to the long-term possibilities. Empirical studies of FDI spillover effects on domestic firms reflect various factors, conditions, and characteristics of the firm, at industry and national levels. The reported results do not reproduce the ambiguous effects of economic sectors on labour productivity, or undervalued labour costs per worker, and do not consider the role of the shadow economy in the countries of Central, Eastern and Southeast Europe. Inadequate skills and education of workers have been a major or severe obstacle for multinational firms' operations in many developing countries.

The government policy of FDI inflow liberalisation induces incentives for foreign companies to enter the local markets of other countries. Government support for education and training, and the realisation of benefits arising from FDI have prompted governments to encourage FDI inflow. This paper discusses the challenges faced by governments in creating policies to attract FDI practices in emerging economies.

The inflow of FDI stimulates the development of modern technologies, management, marketing, and the use of advanced methods of labour organisation. The FDI inflows affect and promote labour productivity changes. The higher wage rates lead to rising aggregate demand. The higher investment, along with total productivity improvements, could reinforce the current account position. The increasing technological transparency of the information society emerging in European countries, as well as growing intra-European bilateral FDI links, have contributed to a greater incidence of technology spillovers and external scale economies.

Globalisation has affected the creation and enlargement of the world's supply of a relatively unskilled labour force. As a result of it, international trade for new products rises. It stimulates the redistribution of the labour force from high skilled sectors to unskilled sectors. There is a growing number of unskilled workers in East European countries. The increase in unemployment rates in market economies is combined with a large share of concealed unemployment in East European countries and this calls for government interventions in labour markets. It is important to mention the role of the government as the political institution that should correct market forces to provide adjustments in the institutional framework. The penetration of transnational corporations (TNCs) into the domestic market accelerates product development, creates new working places, brings in new management of the organisation and improves the welfare of workers. The problem of the relationship between labour quality and labour cost has been a subject to many scientific works.

Studies indicate that low wages in developing countries attract FDI. Nevertheless, there are surveys that reveal most FDI occurs between countries with similar wages [1, p. 123]. Many low wage countries have the lowest levels of FDI. The COVID-19 crisis negatively affected the movement of international capital. In the first half of 2020, global FDI flows fell by 50% compared with the last half of 2019, to $364 billion, because of the pandemic and the resulting supply disruptions, demand contractions and pessimistic outlook of economic actors [2].

The physical closure of places and construction sites resulted in a reduction in international production, and disruption of supply and distribution chains. Delays of greenfield projects, and the reduction in mergers and acquisitions (M&A) have prevented FDI inflows rising, and this highlights the negative consequences of the pandemic's effects on international investment policies.

Foreign competitors stimulate local firms to compete for more efficient production in the industry. Furthermore, after the period MNCs running their business in the country, they bring positive impacts on downstream markets.

The paper devoted to analysis of FDI effects on the productivity and local firms, government policy for an attractive investment climate, and FDI spillover effects on domestic firms across countries. The world data on capital movement demonstrate fluctuations and unpredictability in the global economy aggravated by wars in Ukraine and Israel and by the environmental crisis. The goal of the article is to analyse the relationship of transnational corporations (TNCs) and FDI, determine the effects of FDI on productivity and local firms, to analyse inflow, outflow, and net flow FDI, impact FDI inflows on TNCs activity, FDI in startups, examine national investment policies.

1. Literature review

Scientists consider capital movement in the form of FDI may increase opportunities for investment attraction and creates challenges for economies. The impact of FDI on economic growth was investigated empirically by Hassouba et al [3] and they found that FDI and cross-border capital inflows had a positive impact on growth. According to Ariyani and Firmansyah [4], FDI is positively correlated by market size, anti-corruption measures, and telecommunications infrastructure, and negatively affected in selected Asian Emerging Market countries. As a result, it causes low wages for labour and this is still the target of foreign investors, compared to skilled workers with better levels of education but higher wages. Bardesi [5] assesses how FDI affects job growth in domestic businesses involved in related industries and considers the increase in employment.

The effects of FDI on host countries' economies are related to increasing labour productivity through technological transfers, and management and marketing proficiency that enables long-term technological progress and economic growth. Estrin et al [6] have presented that the development of performing management skills according to the standards imposed by the major leading corporate systems and increasing the population's training level and its capacity to adapt to the technological developments can contribute to increasing the quality of labour resources.

Projects attract bank credits for fixed capital investments. An examination of the range of opinions leads to a conclusion on the prevalence of capital formation as one of the financial sources of fixed capital formation abroad, namely debt financing, capital market financing and subsidies. Krkoska [7] argues that capital formation is positively associated with FDI, along with domestic debt and capital market financing, but negatively correlated with stock market liquidity.

The data of contribution to global growth depict that in the forecast of world growth in 2024 will decrease to 2.4% related to 3.1% in 2010-2019, in the USA the growth contribution will make up 0.2%. The statistics in the Euro area will be without changes and will make up 0.2%, and 0.7% in emerging market and developing economies [8]. This slowdown reflects the current trend of falling global potential production growth as well as cyclical dynamics.

Cieslik and Ryan [9] found that corruption hinders a country's ability to draw investment and that poor countries are more severely affected than industrialized ones. They also observe that corruption has different effects in different businesses. In developing and transitioning economies, corruption has a bigger impact on manufacturing than on services, and within manufacturing, it has a bigger impact on investments in companies that make electrical equipment and machinery.

Empirical research of Sanusi and Eita [10] illustrates that trade openness and human capital have a favourable and significant impact on economic growth. FDI, however, could be detrimental to economic expansion. Munene [11] applied an autoregressive distributed lag (ARDL) model and discovered that FDI and trade openness have a long-term favourable impact on economic growth. In the short-term findings, trade openness had a favourable and significant effect on FDI [11]. However, the effect of FDI on economic growth in the medium term is not statistically significant. Like FDI effects, economic growth had a minor short-term influence but a positive long-term effect. In the context of global instability, Stefaniuc and Biloocala [12] pinpoint the contemporary tendencies of FDI flows. The global economy's increased uncertainty and crisis caused by financial issues are the cause of fluctuations in FDI flows. Financial globalization and its intensification have led to an increase in global FDI volumes. The global FDI market is becoming unstable and cyclical due to international investment cooperation and growing instability. Nguyen [13] assesses how monetary policy affects FDI attraction. The study's findings support the idea that while contractionary monetary policy has helped Southeast Asian nations attract more foreign direct investment, expansionary monetary policy has the opposite impact [13]. The study also supports the beneficial effects of trade liberalization and high human resource standards on a country's capacity to draw in foreign direct investment.

To ensure the sustainable development of a country, it is necessary to ensure the growth of FDI inflow. Foreign affiliates influence the export opportunities of domestic firms through trade, which helps to reduce the costs of domestic firms in penetrating foreign markets. FDI and products of foreign affiliates are substitutes for exports, as well as complements to them.

2. Methodology

The article applies a comparison of economic methods, defining the object of the study in relation to other approaches and methodological tools. It is an analytical technique used in social sciences to define, categorize, and interpret data by looking for patterns of similarity and difference among the numerous things being studied. The article analyses the beneficial effects of trade liberalization and human resource inflows on a country's capacity to draw in FDI. The study evaluates the application of comparative studies, explores their use and interpretation, and then introduces the analysis of comparative data.

Fig. 1 illustrates Foreign Direct Investment Flows in US Millions of Dollars in the period from 1990 to 2020 [14]. Fig. 2 displays Changes of Global Foreign Direct Investment by Countries in 2022 compared to 2021 [14]. Fig. 3 illustrates changes in national investment policies from 2013 to 2022 [14]. Table 1 represents the data of Net annual Foreign Direct Investment flows [15].

3. Discussions

Empirical surveys of the international organisations United Nations Conference on Trade and Development (UNCTAD), World Bank, International Monetary Fund (IMF) and WIFO (Austrian Institute of Economic Research) have pointed out that the basic volume of the cross-sectional movement of capital flows is carried out in the form of FDIs. Positive effects of this on the economy of the recipient country include: an increase in the volume of real capital investments, an acceleration in the pace of economic development, an improvement of the country's balance of payments, receipt of advanced foreign technology, organisational and managerial experience, and the results embodied in new technology, patents, licences, and know-how. Foreign capital inflow increases the level of employment and qualifications of the local labour force, raising the produc-

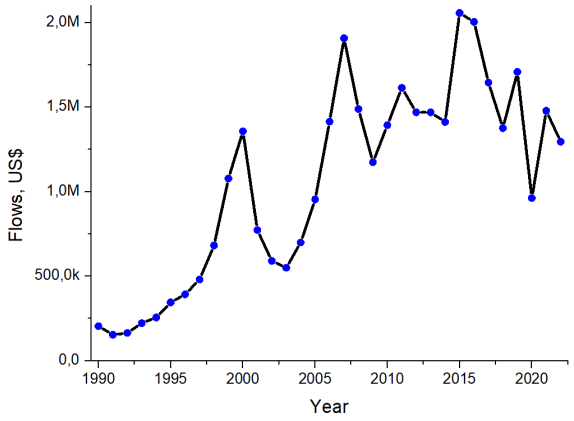


Fig. 1. Foreign Direct Investment Flows (US Million$). Adapted according data of Ref. [14].

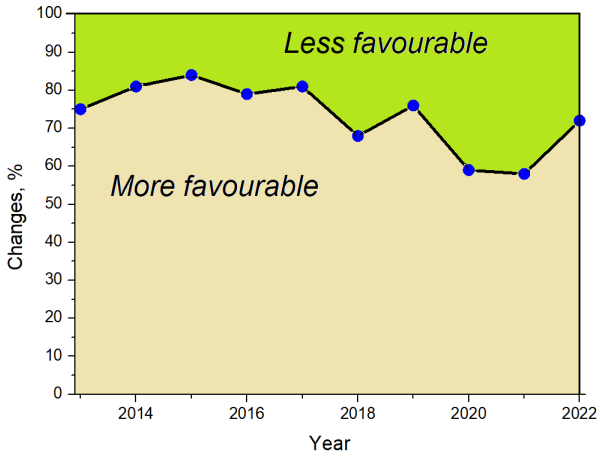


Fig. 3. Changes in national investment policies, in %. Adapted according data of Ref. [14].

tivity of the labour, and improves the standard of living and purchasing power of the population. Changes in skill structures and job characteristics demonstrate a cautious upturn in economic activities in East European countries. This region has maintained its cost competitiveness, despite surging wages and occasional labour shortages, by benefiting from considerable productivity improvements an upturn over the last decade [16]. Negative consequences of foreign capital inflow include suppressing local producers and limiting competition, repatriation of capital and transfers of profits in various forms (dividends, interest, royalties) which worsens the balance of payments, increasing dependence of the national economy on foreign states which threatens the economic and political security of the recipient country, and ignoring local conditions and peculiarities by foreign investors.

Questions about the size of FDI's role in the process of capital formation and total investment escalation have received ambiguous consideration in the economic literature. FDI inflows did not provide the creation and return of fixed capital. The purchase of a company by a foreign investor leads to a change in property relations.

With the help of FDI, countries encourage resource spending and inflow of finished goods. Domestic companies import technologies from MNCs through the purchase of production equipment, industrial capacity, and differentiated products. The impact of FDI on host country economy varies by distinguishing between the growth effects of horizontal (market- seeking) FDI and vertical (efficiency-seeking) FDI inflows. Beugelsdijk et al [17] argue that horizontal and vertical FDI have positive and significant growth effects in developed countries. The authors indicate a superior growth effect of horizontal FDI inflows over vertical FDI inflows in transition economies. They estimate the absence of any significant effects of horizontal or vertical FDI inflows in developing countries. FDI flows in transition economies in South-east Europe and CIS declined by 27% in 2017, to $47 billion, the second lowest level since 2005. Outflows from transition economies rose by 59% to $40 billion [16].

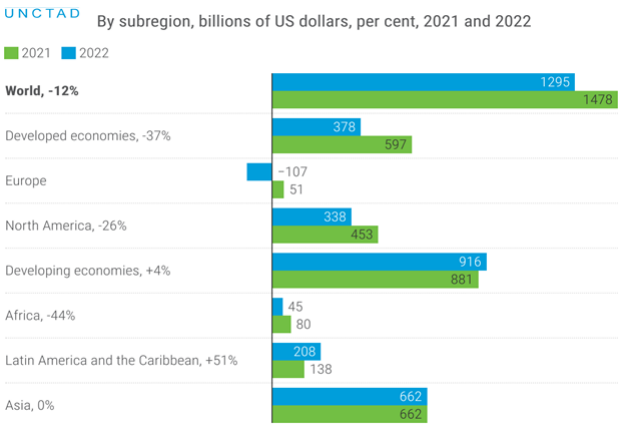


Fig. 2. Changes of Global Foreign Direct Investment by Countries in 2022 compared to 2021, in %. Adapted according to Ref. [14].

The UNCTAD data for global FDI flows in 2020 demonstrate a decrease by up to 40%, from a value of $1.54 trillion, and projects a further decline by 5–10% in 2021 [14], [18, p. 8] as presented in Fig.1. This figure indicates the data of direct investment inflow from 1990 to 2021. FDI inflow curve shows wave-like jumps in investment volumes with downturn tendency in 2020. The graph confirms the decreasing tendency for global FDI flows in 2020.

The unstable situation with the COVID-19 pandemic in the world, ineffective measures to curb the development of the virus, disruptions in the supply-chain, the introduction of tough conditions in financial markets, and changes in human needs and behaviour at the markets have forced investors to delay the implementation of international projects. The reduction in production caused by the emergency in the health care system has led to greater losses of human lives. The unpredictability has affected the global economy in the acceleration of instability of international investors' economic behaviour. Uncertainty also reduces demand in the private sector, increases households' savings and reduces their expenditures [19, p.76]. Economic policy plays an active role in stimulating aggregate demand. The adoption of support measures for the economic development of the most vulnerable sectors in the pandemic, such as services (tourism), restaurants, entertainment, and construction, provides the basis for protecting workers in affected sectors.

On average, the top five thousand multinational enterprises (MNEs), which account for a significant share of global FDI, have seen downward revisions of 2020 earnings estimates of 9% due to COVID-19. Hardest hit is the automotive industry (≈44%), airlines (≈42%), and energy and basic materials industries (≈13%). Profits of MNEs based in emerging economies are more at risk than those

Table 1. Net annual Foreign Direct Investment flows (in million $US). Constructed using data of Ref. [15].

	2014	2015	2016	2017	2018	2019	2020	2021	2022
Hungary	0.00	0.00	0.00		0.50	0.00	0.08	-0.20	0.02
Poland	0.05	0.13	0.36	0.14	0.22	0.65	0.22	0.05	0.29
Slovakia	0.00	0.00	0.32		0.01	0.01	-	0.03	-
Slovenia	0.00	0.00	2.40		0.23	0.22	0.01	0.00	0.14
Czech Republic	-	0.09	10.09	0.11	0.07	0.11	0.06	0.12	0.17
Russian Federation	0.00	0.05	0.01	0.00		0.11	0.11	0.21	25.43
Ukraine	0.00	0.00	0.00	0.00	0.00	0.01		0.01	0.00

of MNEs in developed countries; profit guidance for the latter has been revised downwards by 16% [18, p. 1].

For analysis Dingel and Neiman [20] consider non-standard forms of work and include part-time workers, the self-employed, and workers hired on fixed-term contracts. The authors provide estimates of the share of non-standard workers who are particularly vulnerable to the loss of income or jobs because of the impact of COVID-19. Their study relates labour market disruptions and discusses what policies can do and what policy actions governments have taken, to support vulnerable workers and promote an inclusive labour market recovery. Cavalleru and Cuasa [21] provide estimations of patterns and trends in socio-economic groups. In the sectors most directly affected, non-standard workers represent around 40% of total employment on average across European countries, ranging from about 20% in Latvia and Lithuania, to more than 50% in Italy, the Netherlands, Spain, and Greece.

After a decline of 37% in 2019 (to $24 billion), outward FDI from economies in transition expected to continues to decline in 2020 and 2021, as economic recessions in home economies and low oil prices affect the capacities of MNEs from the region to invest abroad. FDI flows to the transition economies of South-east Europe, the Commonwealth of Independent States (CIS) and Georgia, hit hard by the economic downturn caused by COVID-19, declined by around 38% in 2020. These flows will not recover before 2022, according to UNCTAD data [22, p. 56].

Global FDI fell by 12% in 2022 to $1.3 trillion after a sharp dip in 2020 and a robust recovery in 2021. The global poly crisis, which includes the conflict in Ukraine, rising prices for food and energy, and debt pressure, is what caused the slowdown. Greater financial constraints, rising interest rates, and uncertainty in the capital markets had a particularly negative impact on international project finance and cross-border mergers and acquisitions (M&As) [14]. Fig. 2 demonstrates fell of global FDI on 12% in 2022 compared to 2021 and decreases on 37% in the developed economies.

Greenfield investments are a type of FDI where a company starts its operation in other countries as its subsidiary and invests in the construction of offices, plants, sites, building products, etc., thereby managing its operations and achieving the highest level of the controls over it. In a green-field project, a company plant construction, for example, is done to its specifications, employees are trained to company standards, and fabrication processes can be tightly controlled. As the most widespread form of FDI inflow in less developed countries, "zero" investments (greenfield investments) act which made in the form of new enterprises establishment and promote expansion of a company's capacities in comparison with the acquisition process of existing companies. New investments accelerate economic growth by increasing supply both nationally and in terms of the companies that are controllable by foreign proprietors under liberalised trade condition in the country [23]. Based on the previously mentioned facts, it is necessary to note that growth of the international movement of capital scales is accompanied by amplification of international TNCs expansion in transition countries. The source of savings growth is FDI, borrowing from international organisations, assistance from developed countries. The growth of the domestic savings affects inflow of foreign direct investment and stimulates the economic growth in the country.

The comparison of gross savings and savings rates emphasises differences between countries. Misztal [24] highlights a positive correlation between economic growth and savings rates. In the short run, a growth rate of savings leads to a quantitative rise in production. In the long run, this process is accompanied by increases in capital intensity and in the volume of produced output [24].

The development of privatisation and commercial activity opened and expanded access for foreign capital to the new markets and stimulated the creation of new manufacturing branches in countries that were recipients of foreign capital. In Table 1, data of the net annual FDI inflow/outflow in transition countries during the period from 2014 to 2022 is shown. Data show a significant increase in net annual FDI inflows in Russian Federation in 2022 that confirms the return of significant flows of Russian offshore capital back to the country.

Inbound FDI to economies in transition increased sharply in 2019 (by 59% to $55 billion), due to higher inflows in large countries, especially the Russian Federation, Ukraine, and Uzbekistan. Flows to the rest of the region declined slightly (down 3% to $19 billion). In Ukraine, FDI flows rose by 30%, to $3.1 billion, after two years of decline [22, p.56]. Finance, ICT, mining, real estate, and electricity and gas attracted the bulk of FDI.

A significant share of FDI, at about one-third of the value in 2020, is estimated to be round-tripping of the Ukrainian capital through offshore centres. Market-seeking projects will also suffer in that country and in others in the region as the economic downturn deepens. Foreign affiliates are facing exceptionally challenging operational, market and financial conditions. Their profits are expected to plummet in 2020. FDI outflows from Ukraine reached 0.6 billion in 2019 [22, p.22, p.57]. Based on the above mentioned, we presume that international capital movement is accompanied by increasing international penetration and expansion of TNCs in transition countries. The internationalisation rate of companies from developing to transition economies increased by almost 2%, with foreign assets and sales growing faster. Employment by foreign affiliates increased from 27729 in 1990 to 82360 in 2019 [22, p.8, p.24]. The pandemic and low oil prices have affected FDI flows.

Estimation of the pandemic's impact on global capital movements has drawn the attention of scientists towards the consequences resulting in recession in economies, production or supply chain disruption, a reduction in employment and an increase in unemployment, and an increase in inequality.

Trade policies can stimulate exports and imports, especially of intermediate and capital goods, which can lead to gains in productivity. In addition, encouraging firms to innovate and conduct research and development activities through fiscal incentives and financial benefits that are aimed at making industry (in particular, manufacturing) and services more efficient, technologically up to date and

competitive, can also lead to growth in productivity [25].

Domestic companies import technologies from MNCs through the purchase of production equipment, industrial capacity and differentiated products. Defining the internal effect, Jude and Levieuge [26] investigated the impact of FDI on economic growth in developing countries and concluded that the institutional quality modulates the impact of FDI, while a favourable institutional environment induces a growth-enhancing effect. The external effect of the interaction of foreign branches and domestic firms on horizontal or vertical levels contributes to increased productivity in the country. Growing demand for intermediate products is forcing domestic firms to take advantage of economies of scale. Determining the impact of FDI on a country's economic development has made it possible to identify technology transfer through foreign affiliates as the main driver of economic growth. Competition development among domestic firms, productivity increases, the construction of new organisational and production structures, and the introduction of new technologies attract FDI inflow [27, p.20].

Authors have shown that the effect of privatization is mostly positive in Central Europe, but quantitatively smaller than that to foreign owners and greater in the later than earlier transition period. In the Commonwealth of Independent States, privatization to foreign owners yields a positive or insignificant effect while privatization to domestic owners generates a negative or insignificant effect. Through technology transfer and technology spillovers FDI can facilitate international collaboration on R&D.

MNCs try to place labour-intensive products in Eastern Europe and Asia, where wages and units of labour costs are cheaper than in Western Europe. Wage pressures are encouraged by increasing capital intensity and the need to increase the volume of R&D. These measures improve the quality of products and force companies to maintain higher prices in world markets. The introduction of modern technologies requires skilled workers, as they cope better with technological change.

This is true for diverse types of firms and various levels of technological development. Technology transfers by multinational firms and the application of technology by local firms require the use of a minimum of human capital and training of a skilled workforce. The use of modern technologies usually requires significant organisational changes of companies. MNCs are applied better strategies by the attraction of a skilled labour force. Lack of employees with higher education can be a more deterrent for firms in production and value-added services than for less complex production processes.

The lack of skilled workers is a widespread problem for MNCs in developing countries. This is especially so for companies that plan to innovate and expand their scale of production. World Bank data show that firms that consider the shortage of skilled workers to be a "major" or "very serious" constraint are those that improve their production processes. These firms are also more likely to invest in training their workforce. Smaller companies often do not offer internal training programs to their employees, although larger companies do. [28, p.136-137].

Startups that embrace innovation and have the potential to upend established industries tend to draw investors. Startups encourage their staff to think creatively and come up with ground-breaking solutions by fostering an innovative culture within the company. When a startup demonstrates that it is committed to keeping ahead of the curve and improving continuously, it might attract overseas investors looking for innovative firms. FDI attraction creates new opportunities for firms trying to grow and seize new markets. Startups can greatly increase their chances of attracting FDI by creating a compelling value proposition, demonstrating market potential, establishing a track record of success, utilizing networking and connections, participating in government initiatives, creating a thorough business plan, and encouraging an innovative culture. The example of successful startups could be analysed in India. Significant foreign ownership is present in the well-known Indian startups *Flipkart*, *Snapdeal*, *Ola*, *Zomato*, *Swiggy*, and *Paytm*, which is not a positive sign. These businesses, each founded by a different person, are estimated to be worth over $1 billion. Indian startups attract the interest of foreign investors due to their immense potential to yield substantial returns and rapid expansion. All startups have a different model and various structures, invest significant expenditures in marketing and human resources.

Global venture capital (VC) investment in 2021 totalled a record $687bn, up from $353.3bn the previous year, marking a growth rate of almost 95%, according to US Venture Capital Outlook data [29]. Global VC in 2021 was ten times higher than a decade earlier [30].

Fig. 3 illustrates changes in national investment policies from 2013 to 2022 and proves that after COVID-19 there were some improvements in policies for favourable investment in the world. A healthy investment climate increases the incentive for people to attain a higher level of education. The investment in human capital is the indication of the large increase in income from education in the former centrally planned economies during their transition to market systems. Similar patterns have emerged in other countries. A high level of formal education is not required for all firms or activities. A lack of employees with higher education can be more of a deterrent for firms in manufacturing and value-added services than for less complex production processes.

The allocation of a large stock of MNC's foreign investment abroad may lead to its relocation in the event of political and economic threats. The parent company will stimulate FDI outflows and capital transfers to new locations. Multinational firms use a model of negotiation abroad that they are familiar with in terms of international relations (IR) and international business (IB) and reconstituting intellectual boundaries. Highlighting two approaches, Jarvis [31] suggests that the construction of new interdisciplinary rubrics jointly created from IR and IB offers a better means of appreciating the changing character of the global political economy, and some of its most important actors and emerging processes [31, p.220].

The further study FDI spillover effects consider the assessment of an increase in the productivity level of local firms and competition in Eastern Europe. The actual relationship between horizontal and vertical FDI spillover effects remains unclear, although the available research has identified positive correlations. The use of the comparative analysis method for the practice of attracting foreign capital provides a sound policy and an appropriate analysis of crucial challenges to encourage FDI inflow.

Spillover effects comprise technology transfer, labour, and management training. Two indirect effects of the presence of foreign capital take places externally, to other firms in the host economy. Direct effects of FDI result in economic growth. Indirect effects of FDI within the host economy take the form of horizontal or vertical spillovers of capital flows. In this case, domestic firms occupy adequate positions in the production chain with foreign affiliates, competing with them. They force less efficient firms to close. The average productivity of the industry in the host economy will rise, which in turn stimulates productivity growth within firms.

An evaluation of various empirical studies of FDI spillover effects on domestic firms reflect various factors, conditions, technologies, products and characteristics at the firm, industry and national levels based on econometric assessments is discussed in further research.

Arif-Ur-Rahman and Inaba [32] assessed FDI spillover on firms' productivity in Bangladesh in comparison to Vietnam. The authors consider that Vietnamese firms benefit from backward linkages while Bangladeshi firms receive profit from intra-industry or horizontal links. They considered that increases in foreign presence in downstream industries for Vietnam and Bangladesh are associated to rises in domestic firms' output [32].

Estrin believes that the inflow of FDI is associated with rising GDP and declining unemployment in general, as well as over time in most regions [33, p.2]. The effects were most noticeable and marked earlier in the EU, and least in the former Soviet republics and Russia. Earlier in the EU's history FDI affected natural resources in Russia and some Central Asian republics, and the impact on employment was less determined. The author emphasises that the indirect impact of FDI on the restructuring of a company, its productivity and employment were very significant in the EU member states. Insufficient development of institutions, and the gap between investment needs and domestic capabilities of companies hinder the attraction of foreign capital.

Fillat and Woerz [34] argue that FDI depends on additional factors in order to exert a significant effect on growth, i.e. a significant level of domestic investment or export orientation is necessary [34, p.320]. Using a comparable database at the industry level for 35 countries in the OECD and Eastern Europe from 1987 to 2002, the authors test for the influence of both the stage of development and sectoral FDI patterns in the relationship between FDI and productivity growth. The authors argue that a significant and positive relationship emerges when FDI coincides with high investment or the export orientation of a country.

The study of the previously mentioned approaches allows to identify the main factors stimulating economic growth and technology transfer through the placement of branches of foreign firms, the development of competition with domestic firms and increasing productivity by attracting FDI [35, p. 124].

Empirical studies of FDI spillover effects on domestic firms across countries confirm the existence of direct and indirect effects, and reflect various technologies, factors of production. The reported results do not reproduce different effects of economic sectors, labour productivity, or undervalued labour costs per worker. Moreover, if internal and external effects act in the same direction, reducing labour costs per unit of output. They perform as a factor stimulating the efficiency rise, output increase, and product's quality and competitiveness improvement.

Conclusions

FDI effects on the productivity and local firms show ambiguous trends that depend on the country's economic development, investment in R&D and reduction of production costs. The decision to invest FDI depends on the assessment of macroeconomic factors and the investment attractiveness of the country.

Economic growth will boost by a policy to draw in foreign capital to stimulate both internal and external benefits of FDI on labour productivity. Improvement of the investment climate goes hand in hand with enlargement and development of human capital. A skilled workforce is important for firms using new and productive technologies. An attractive investment climate increases returns on investment in human capital.

FDI attraction creates new opportunities for innovative firms trying to grow and seize new markets. Startups can greatly increase their chances of attracting FDI by creating a compelling value proposition, demonstrating market potential, establishing a track record of success, utilizing networking and connections, participating in government initiatives, creating a thorough business plan, and encouraging an innovative culture. The example of successful startups in India is analysed. Governments need to take the initiative to enhance the development of education, to make it more comprehensive and responsive to the needs of business.

Wage growth at overseas affiliates is stimulated by the expansion of foreign capital. Compared to domestic companies, workers in industries with higher foreign participation receive faster wage increases. Rising productivity and raising the standard of life of the populace in nations that receive foreign investment are the goals of creating an attractive investment climate, adjusting legislation, and incentive programs to create new jobs.

Abbreviations

CIS	-	Commonwealth of Independent States
EBRD	-	European Bank for Reconstruction and Development
FDI	-	Foreign Direct Investment
GDP	-	Gross Domestic Product
ICT	-	Information and Communication Technology
IMF	-	International Monetary Fund
IB	-	International Business
IR	-	International Relations
M&A	-	Mergers and Acquisitions
MNCs	-	Multinational companies
MNEs	-	Multinational Enterprises
OECD	-	Organization for Economic Cooperation and Development
R&D	-	Research and Development
TNCs	-	Transnational companies
UNCTAD	-	United Nations Conference on Trade and Development
USA	-	United States of America
WIFO	-	Austrian Institute of Economic Research

References

1. UNCTAD (2001) World Investment Report: Measures of the Transnationalization of Economic Activity. 2001 - United Nations, New York, and Geneva. - https://unctad.org/ru/docs/wir01ove.ru.pdf.
2. OECD (2020) FDI in Figures. COVID-19 Disruptions Send Global FDI Plunging 50%. 2020. - http://www.oecd.org/investment/statistics.htm.
3. Hassouba, T., A., A.; Aboeta, M., M., A. (2023) The Impact of Foreign Direct Investment on Economic Growth: Empirical Evidence from Egypt (1980-2022). - *Journal of Alexandria University for Administrative Sciences* 60(2023).
4. Ariyani, A., D.; Firmansyah, F. (2023) Determining factors of foreign direct investment in Emerging Market Asia: A panel data analysis (2005-2020) Determining factors of foreign direct investment in Emerging Market Asia: A panel data analysis (2005-2020) - *OPTIMUM: Journal Ekonomi dan Pembangunan* 13(2023)195-206 - https://dx.doi.org/10.12928/optimum.v13i2.8535.
5. Bardesi, H. J. (2023) Foreign Direct Investment and Job Creation in Saudi Arabia - *International Research Journal of Economics and Management Studies* 2(2023) 375-387 - https://doi.org/10.56472/25835238/IRJEMS-V2I3P150.
6. Estrin, S.; Hanousek, J.; Kocenda, E.; Svejnar, J. (2009) The effects of privatisation and ownership in transition economies. - *Journal of Economic Literature* 47(2009)699–728 - https://doi.org/10.1257/jel.47.3.699

7. Krkoska, L. (2001) Foreign Direct Investment Financing of Capital Formation in Central and Eastern Europe (December). - *EBRD Working Paper* 67 (2001) - https://ssrn.com/abstract=314270.
8. Global Economic Prospects (2023) International Bank for Reconstruction and Development - The World Bank. Washington D.C.
9. Cieslik, A.; Ryan, M. (2023) Host Country Corruption and Japanese Outward Foreign Direct Investment: A Sectoral and Industry. Level Analysis. - *Journal of Multinational Financial Management* 70-71(2023) - https://doi.org/10.1016/j.mulfin.2023.100820.
10. Sanusi, K., A.; Eita, J., H. (2023) Trade Openness, Foreign Direct Investment and Economic Growth: Case of Southern African Development Community - *Cuadernos de Economia* 46(2023)64-74.
11. Munene, G., E. (2023) Foreign Direct Investment, Trade Openness and Economic Growth in Kenya: Empirical Analysis Using ARDL Approach. - *International Journal of Science and Business* 28(2023)115-126.
12. Stefaniuc, O.; Biloocala, S. (2023) The Modern Tendencies of Foreign Direct Investment Flows in Conditions of Global Instability. - *Studia Universitatis Moldaviae Revista științifica a Universtității de Stat din Moldova* 11(2023) - https://doi.org/10.59295/sum11(01)2023_07.
13. Nguyen, V., C. (2023) Monetary Policy and Foreign Direct Investment – Empirical Evidence. - *Economies* 11(2023) - https://doi.org/10.3390/economies11090234.
14. UNCTAD (2023) Investing in Sustainable Energy for all. 2023 United Nations, New York, and Geneva - https://unctad.org/publication/world-investment-report-2023.
15. World Bank data, Net Foreign Direct Investment Flows (BPM 6) (2020) - https://www.bsp.gov.ph/statistics/external/tab10_fdc.aspx.
16. UNCTAD (2018) World Investment Report: Investment and New Industrial Policies. Key Messages and Overview. - United Nations, New York, and Geneva - - https://unctad.org/en/PublicationsLibrary/wir2018_en.pdf.
17. Beugelsdijk, S.; Smeets R.; Zwinkels, R. (2008) The impact of horizontal and vertical FDI on host's country economic growth - *International Business Review* 17(2008)452-472.
18. UNCTAD (2020) Investment Trends Monitor. Impact of the Coronavirus outbreak on global FDI. - United Nations, New York, and Geneva- https://unctad.org/system/files/information-document/diae_gitm34_coronavirus_8march2020.pdf.
19. Paskeleva, M. (2020) Global Uncertainty During COVID 19 Pandemic and Its Effects on EU Member States with A Focus on Bulgaria - *International Conference Knowledge-Based Organization* XXVI(2020)71-76 - https://doi.org/10.2478/kbo-2020-0055.
20. Dingel, J.; Neiman, B. (2020) How many jobs can be done at home - *Journal of Public Economics, Issue C.* 189 (2020) - https://econpapers.repec.org/article/eeepubeco/v_3a189_3ay_3a2020_3ai_3ac_3as0047272720300992.htm
21. Cavalleru, M. C.; Cuasa, O. (2020) How government can protect vulnerable workers - https://www.weforum.org/agenda/2020/07/how-non-standard-workers-are-affected-and-protected- during-the-covid-19-crisis-stylised-facts-and-policy-considerations/
22. UNCTAD (2020) World Investment Report: International Production Beyond the Pandemic - - United Nations, New York, and Geneva - https://unctad.org/system/files/official-document/wir2020_en.pdf.
23. Graham, E. M. (2000) Fighting the wrong enemy. Antiglobal activists and multinational enterprises. - Institute for International Economics, Washington D.C, September. 2000.
24. Misztal, P. (2011) The relationship between savings and economic growth in countries with different level of economic development. - University of Information Technology and Management, Rzeszów - *Financial Internet Quarterly* 7(2011)17-29.
25. Sharma, C. (2010) Does productivity differ in domestic and foreign firms? Evidence from Indian machinery industry. - *Indian economic review* 45(2010)87–110.
26. Jude, C.; Levieuge, G. (2014) Growth Effect of FDI in Developing Economies: The Role of Institutional Quality. - *Working Papers* (2014) - https://halshs.archives-ouvertes.fr/halshs-01014404.
27. Nosova, O. (2016) Effect of Attracting FDI to Economic Growth of TNCs. - Management and Venture Capital of Syktyvkar State University. - *Bulletin of the Research Center of Corporate Law* 4(2016)13-23.
28. UNCTAD (2006) World Investment Report. FDI from Developing and Transition Economies: Implications for Development - United Nations, New York, and Geneva - https://unctad.org/system/files/official-document/wir2006_en.pdf.
29. US Venture Capital Outlook (2023) Pitchbook. December 18. - https://pitchbook.com/news/reports/2024-us-venture-capital-outlook.
30. Hunt, I. A. (2022) Global Venture Capital Fuelled Investment Slows. - https://www.fdiintelligence.com/content/data-trends/global-venture-capitalfuelled-investment-slows-80677.
31. Jarvis, D., S., L. (2005) Multinational enterprises, international relations and international business: reconstituting intellectual boundaries for the new millennium. - *Australian Journal of International Affairs* 59(2005)201-223 - https://doi.org/10.1080/10357710500134459.
32. Arif-Ur-Rahman, M.; Inaba, K. (2021) Foreign direct investment and productivity spillovers: a firm-level analysis of Bangladesh in comparison with Vietnam. - *Economic Structures* 10(2021)17 - https://doi.org/10.1186/s40008-021-00248-2.
33. Estrin, S. (2017) Foreign Direct Investment and Employment in Transition Economies. - *IZA World of Labor* (2017) 330 - https://doi.org/10.15185/izawol.330.
34. Fillat, C.; Woerz, J. (2011) Good or Bad? The Influence of FDI On Productivity Growth: An Industry-Level Analysis. - *The Journal of International Trade & Economic Development. An International and Comparative Review* 20(2011)293-328.
35. Nosova, O. (2014) Investment Perspectives of Ukraine: Economic and Legal Model. - Monograph - LAPLAMBERT Academic Publishing, 2014.

Review article

Electronic Voting System. History, Problems, and Perspectives

Vytautas Šulus*
Department of IT, Vilnius Business College, Kalvarijų g. 129-401, LT-08221 Vilnius, Lithuania
* Corresponding author, e-mail: *vytautas.sulus@gmail.com*

Received: 15 October 2023
Accepted: 15 December 2023
Online: 30 December 2023

JEL: D72.

Abstract. This paper presents a theoretical analysis of the development and application of electronic elections (e-elections) in the context of cyber security management. Components and tools of e-elections were presented and discussed in the organizational framework. The implementation of e-elections was analysed on the example of Estonia and Switzerland.

Keywords: e-elections; e-voting.
Citation: Vytautas Šulus (2023) Electronic Voting System. History, Problems, and Perspectives. – *Applied Business: Issues & Solutions* 2(2023)10–21 – ISSN 2783-6967.
https://doi.org/10.57005/ab.2023.2.2

Introduction

During regular elections in Lithuania, voting activity has been decreasing every decade. The highest voter turnout in Lithuania was recorded in 1992 (about 75%, the first elections to the Seimas after the Restoration of Independence). After 24 years, in 2016, voter's activity was significantly lower (about 50 %, elections to the LR Seimas) [1]. The main reasons for the decline in voter turnout are the aging of the population, Lithuanian emigration, and changing consumer behaviour. About 2.8 million people live in Lithuania. Future forecasts show that by the 2030s, the working-age population of Lithuania will decrease by a quarter [2]. That is, out of 400 thousand emigrating Lithuanians who have the right to vote in 2016, only 17 thousand registered to vote at elections of LR Seimas. Most immigrants and young people do not vote because there is no way to do it electronically [3].

The main reason for the decline in voter turnout is emigration, so electronic voting would encourage Lithuanian citizens living abroad to participate in the elections as well [4]. The change in consumer behaviour due to competition in the electronic space also has a significant impact on voting activity. Society is increasingly inclined to use modern technologies (online elections, ordering services, paying taxes, electronic banking, etc.), which encourages some competitors and state structures to create a convenient and reliable electronic system that would benefit every user.

Establishment of electronic voting in the country could help solve these problems. In 2006, the Seimas supported the e-voting conception, which formulated the goals and requirements for the imposition of online voting, the electronic voting scheme, and requirements to the legal framework. Electronic voting (e-voting) is based on traditional voting systems that use certain online devices that collect, process and store voting results [5]. The Government of the Lithuania Republic initially planned that the online voting system should be created in the second quarter of 2018, and online voting will be possible for the first time in 2019 municipal elections.

However, due to inadequate system security, several illegal hacking of the system could be possible. Such events may aim to falsify election results. This can lead to public uproar and lead to the questioning of the election results" [6]. Lithuania is the leader in the implementation of FTTH and FTTB in the European Union - more than 21% of households here use fibre-optic Internet [7]. Due to well-developed computer communications, the probability of cyber-attacks is quite high. When implementing electronic voting, it is necessary to be ready to defend against cyber-attacks. For that, it would be necessary to examine the current cyber security systems of Lithuania.

This work is intended to analyse successfully operating Internet election systems in the world, for which there are known sources of threats, possible attack vectors, methods, and methodologies. According to this, it is possible to assess the lack of cyber security in Lithuania, when implementing the electronic election system:
1) to assess the methods of ensuring cyber security, management systems and technologies,
2) to determine the extent to which the legal regulation is undefined, which can lead to cyber incidents,
3) to predict the improvement of management systems and e-voting technologies related to the cyber security.

1. E-voting: Aspects of Cyber Security

1.1 Conception of Cyber Security Management

The internal management processes of modern organizations are not possible without information technologies and information systems and the Internet. The Internet has more and more influence on everyday life, including the global economy [8]. Today, everyday education, basic rights, social interaction, and economy require the smooth operation of information and various technologies. The phenomenal development of cyberspace has brought unprecedented economic growth and new opportunities. Modern research and information security took shape after the emergence of computers and the need to manage information and knowledge in the second half of the 20th century.

However, this is the situation and the emergence of new threats. These trends force countries' governments to take measures and create infrastructure that would ensure further economic development, efficiency, and security [9]. The Cyber Security Strategy of the European Union [10] notes that "deliberate and accidental cyber security incidents are increasing at an alarming rate". These incidents can disrupt the provision of basic services (water, electricity, health care and mobility services). Štitilis [11] observes that "the globalization of electronic space has created unprecedented opportunities to commit crimes from any point in the world, where there is Internet and threats in electronic space are not only for individuals for consumers, but also for remote countries".

The concept of cyber security. In the Cyber Security Law of the Republic of Lithuania [12] cyber security is defined as "a set of legal, information dissemination, organizational and technical measures intended to prevent cyber incidents, to detect, analyse, and

react to them, i.e. to manage the information systems in networks ... and restore system after some incidents". In society, the concept of cyber security is often associated with such concepts as electronic information security, information security, network and information security, and information system security.

Network and information security is the ability of networks and information systems to remain with a certain degree of probability free from accidental intrusions or illegal or deliberate actions that would cause damage to stored or transmitted data and related threads. of those networks or systems for the availability, authenticity of the services received, integrity and confidentiality (stored or transmitted data and associated networks). Network and information security ensures a certain level of security, which is used to avoid certain harm. The term does not mean interpretation, analysis, possible response and recovery of activity.

Information security represents the protection of information and system infrastructure from accidental or intentional, natural, or artificial hacking, which can cause damage to the owners of information and system infrastructure. and for users [13]. This concept is only a narrow part of the definition of cyber security, which refers to the protection of information and system infrastructure from possible harm.

The law on cyber security states that cyber security is depending on legal, informational, organizational, and technical tools and services. Jastiuginas [14] defined the conception of information security management in three dimensions.
1. Strategic dimension covers administrative, organizational, government, economic, legal, good tactics aspects.
2. Human factor dimension covers the cultural, ethical, intellectual, educational, psychological, etc. factors affecting safety.
3. Technological dimension includes the aspects of information technologies, technical and software tools, mathematical, cryptographic etc.

Cyber security in the workplace. One of the most important issues when it comes to ensuring cyber security is the question about organization needs: which field must be protected. Countries in European Union (EU) have critical infrastructure that must be protected if events of a cyber incident occur. According to Council Directive [15], two main sectors (energy; transport) and eight subsectors (electricity; oil and gas industry; road, railway, and air transport; inland water transport; sea shipping and seaports) belong to critical infrastructure.

1.2. Essence of Electronic Elections

The penetration of information technologies into various areas of state government is encouraged by the need for transparency, efficiency increase and cost reduction of existing political and administrative processes. "The political and scientific initiatives that have appeared in Europe are examining the implementation of information technologies in electoral processes, health care, tax collection and administration, self-governance, education" [16]. These areas of public education are traditionally criticized for the poor quality of public services, high costs, lack of efficiency, etc.

Current technologies allow automating the execution of tasks in various branches of the industry, where people traditionally play an important role. Electronic voting can be named as one of the representative areas of application of new technologies. The installation of voting systems allowed people to shorten the time of the results announcement, that is, to maximally eliminate the influence of the human factor on the results announcement. Implementation of e-voting and use in various countries showed the high level of requirements which must be placed on the systems to ensure safe, fair, and open elections.

Electronic voting (e-voting) means voting when electronic devices are used in at least one of the polls [17]. Assigned e-voting could be realized in the polling station using devices with touch sensitive screens, standard computers. The results are announced (compressed) after the end of the voting, they can be automatically transferred by network in compressed/encoded form. Each voting data can be recalculated, and the correctness of the voting procedures could be renewed.

Remote e-voting could be realized when voting is done on the Internet, by mobile phone (trial voting only) and at an office, to which a person can go after establishing the agreement of an authorized election commission official or consular office to her. Basic methods of electronic voting [18] are presented below.
1. Voting in the polling station using electronic devices (for example, a computer or a serial voting machine).
2. Unsupervised voting, especially in booths located in public places (trade centres, post offices etc).
3. Internet voting (I-voting) – voting from anywhere using the Internet (computer or phone). I-voting uses information and communication technologies, casting a vote on the Internet.

There is no common definition of the concept of electronic elections in the literature. According to definition of Kiškis et al [16], "Electronic elections are any type of voting that is carried out using electronic devices". Electronic voting can be understood in two ways: elections include the following services: electronic election information, electronic election campaign, electronic election of representatives and electronic referenda [19]. Voting in electronic referenda and electronic elections is the assignment of one's own decision using information technologies.

Electronic elections are defined as any type of voting in which electronic devices are involved in the process. In the context of e-voting, it is important to distinguish the voting using electronic machine and electronic distance voting:
a) for Internet voting, an auxiliary device is necessary which does not determines how voting will be done and allows the voter to vote in a convenient place with an Internet connection;
b) for online voting, an complex system is necessary that includes both Internet voting, voting in the polling station with the help of online voting machines, and online voting in the terminals.

Vaigauskas [20] conducted the voting in an online auxiliary voting system, which can conduct preliminary voting, preserving the traditional voting in the polling station: "everyone can vote in this way who has completed a certain procedure at current time, convinced that he is the voter who has the right to vote. As in case of remote usage of bank service, we sign in using previously declared and documented name and surname, and only after that the system established verified identity" [20].

Paršonis defines this concept differently. In his opinion, electronic voting refers to an ATM, and Internet voting refers to an e-banking. Paršonis claims that electronic voting is a more recent concept that defines the voting model when electronic voting devices are used. This term could be applied on the election day when voting is organized in the polling station using the central voting machines [20].

1.3. Development of Electronic Elections

In 1869 Thomas Edison invented the first electronic voting system and demonstrated this system to the United States Congress. The first proposal for automated voting in Congress was introduced in

1886. Over the next 84 years, more than thirty laws and resolutions were passed to create an automatic, mechanical, and electronic voting systems in Congress. In 1970 a law allowing electronic voting was passed in the USA, and electronic voting was used for the first time in New York in January 23, 1973.

In 1970s, the first system, leading to a computerized electronic voting system, was created by Prof. Murrow Turoffin in New Jersey Institute of Technology (NJIT)[21]. This system named EMISARI (Emergency Management Information System and Reference Index) was developed to organize computer conferences, to discuss topical issues for users and to vote. In 1971 Ohlin [22] described a different system using the terminal connection to the public network. This was supposed to increase people's activism in solving problems related to state governance.

In the middle of the 9th decade of 20th century, many political structures and the public sector more and more often started to organize various projects using Internet network facilities. This was caused by the spread of the global Internet network and improvement of information exchange. Based on these projects, proposals for online elections were presented.

In 1996 Internet voting for the US presidential election was provided [22], where about 2,000 voters voted. In 2000 during the US presidential election, many polling stations refused to use registered ballots. Voting was done in different ways:
a) by manual voting arrows when the respective hand of the selected candidate was turned,
b) by the help of reference cards or electronically, using voting machines.
The first attempt at electronic voting was considered unsuccessful due to the use of different technical equipment. Voting results changed very often, the counting of votes was interrupted. However, experts claimed it happened due to usage of old devices. All recommendations for future (how to improve the processes of voting) were related to replacing the electronic voting with new modern contemporary devices.

In 2002, USA passed the HAVA (Help America Vote Act) [23], which authorized the Election Commission to develop a centralized electronic voter database. The government has decided to allocate budget funds for the purchase of old voting cards and swing voting machines. That is why a commission was created to control the implementation of the election modernization process. in 2006 66 million US voters (38%) had the opportunity to use electronic voting devices. However, 10% of American voters used this option during the elections to the US Congress.

Establishment of electronic voting is actively discussed in other countries as well. In 2002 an initiative group of representatives of the European Parliament submitted a proposal to adopt a resolution on Democracies. The representatives suggested to introduce an online voting system during the elections to the European Parliament in 2004. In their opinion, it could be a promotion of passive voters to express their needs. However, it was decided that such a large-scale project could not be completed in two years. The member states have started to install experimental electronic voting projects in their countries. The United Kingdom had even 150 pilot projects, Switzerland - 8 pilot projects. In 2007 electronic elections were introduced in France, and the introduction of this innovation coincided with a very large number of voters. Several projects related to electronic voting were also implemented in other countries: the Netherlands, Germany, France, UK, Brazil - see Table 1. The pioneer of online elections in the Baltic countries is Estonia, where electronic elections have been in use since 2006.

1.4. Components of Electronic Elections

Stages of voting systems. The Constitution of the Republic of Lithuania and the principles of democracy require ensuring the anonymity and security of voting. The European Security and Cooperation Organization makes such clear demands for democratic elections. The voting equipment used in the election process must ensure a very high level of security due to use the complex set of security protocols.

Basically, any electronic voting system consists of six main elements, which are known as the traditional voting system [24].
1. Voter registration system. The e-voting system provided the voter with authentication data and the possibility to connect to different voting systems. The main task must be the ensuring cyber security: a) for ensuring the transparency of the registration system and b) for protection the system from the unauthorized disclosure of individual information.
2. Authentication. A test intended to verify whether the voter has the right to register and vote. According to the Constitution of the Republic of Lithuania, only persons who have reached the age of majority have the right to vote. Therefore, the register of electronic voter should ensure the legal voting.
3. Voting is the stage when verified voters vote and votes are saved.
4. Management of voting is the stage during which the votes are managed, sorted and prepared for counting.
5. Counting of votes is the stage for decrypting data, counting of votes, and outputting the results.

Table 1. Development of Internet voting: historical facts. Constructed using data of Refs. [22, 23, 24, 25]

Date	Country	Events
1990 - 1999	Belgium	For E-voting, electronic voting machines were used.
1990	Canada	Since then, municipal elections in different cities have been held online using e-voting.
2001	Danmark	Residents of Leeds and Voorburg (Danmark cities) voted online for the new name of the joint city.
2003	UK	An online voting experiment was conducted in the local elections.
2003	Switzerland	In the three cantons, experiment of e-voting was provided.
2004	US	Department of Defence has commissioned a research and development to allow the quality of online voting.
2004 - 2005	Switzerland	Five probing e-voting was held in the cantons of Geneva, Neuchâtel and Zürich (national referenda).
2004 - 2006	Norway	A special commission was instructed to evaluate e-voting opportunities. The two-year commission concluded that e-voting in an uncontrolled region would be possible, but generally it is important to ensure the security aspects.
2005	Estonia	Internet voting was tested and used in local government elections.
2007	Estonia	Internet voting is allowed in the parliamentary elections.
2008	Finland	E-voting was tested in municipal elections.

Table 2. Comparison of the principles of electronic voting

RolNo	Cabello et al [26]	Bogdan et al [27]
1.	Anonymity: it should not be possible to link the ballot to voters	To ensure the anonymity of voting.
2.	Completeness: only those who have a tee can vote to vote.	To eliminate the possibility of vote falsification.
3.	Uniqueness: every legal voter can vote only once.	To ensure the reliability of the software and the save traffic of Internet.
4.	Fairness: every voter should be able to check if your vote has been included	Unique routine of the person identification must be ensured and applied to all voters.
5.	-	To ensure the safety against unauthorized access.
6.	-	To ensure the accuracy of vote counting.
7.	-	The e-voting program must have an intuitive user-friendly interface and must be accessible to people with disabilities.

6. An audit is a test intended to check whether the votes of the voters who voted were included in the final evaluation.

All electronic voting systems have basic minimum requirements that all electronic voting systems must meet. Table 2 represents the main principles formulated by Cabello et al [26] and Bogdan et al [27]. Minimum requirements for e-voting systems were formulated as follows: anonymity, elimination of vote falsification, ensuring the accuracy of vote counting. In e-voting, the provision of personal consent and ensuring the security and reliability of this procedure are of great importance.

Remote authorization available in the following ways:
1) when personal data are confirmed by a qualified electronic signature that meets the established requirements;
2) when personal data are verified by electronic identification devices issued in the European Union, operating under high or low level security electronic identification schemes, which correspond to requirements of Regulation (EU) No. 310/2014 [28];
3) when used electronic devices allow the live video playback in one of the following ways:
3a) during live video translation, the original document confirming the identity or the corresponding permission to obtain in the Republic of Lithuania is recorded and the person identity is confirmed using electronic signature;
3b) during live video translation, capturing the image of the person's face and obtaining the original document confirming the identity or its corresponding permit in the Republic of Lithuania.

1.5. Tools

Use of digital certificate. Identity confirmation or authentication is a method used to verify the source of the transmission or to identify the system participant and to make sure that the transmission has not been modified or hacked during transmission. Personal authentication is a method that allows you to check the authenticity of the subject. In electronic voting, authentication is performed [29] in one of the following routine:
1. Knowledges of subject. The user's account is determined by the password or identification number provided to him.
2. Equipment of subject. Various devices for confirming transactions (magnetic cards, chip cards, generators with a one-time password, etc).
3. Features of subject. Biometric information of a person is available (fingerprints, personal handwriting, retina, etc).

One of the safest ways to do this is to authenticate using a digital certificate. Personal authentication is performed by means of a certification service provider. It is the responsibility of the provider to link the agreement of the person signature with the previous data of signature and verification. Personal digital certificates as the electronic equivalent of a personal contract document are used for this operation. Digital certificate is an electronic equivalent of a driver's license or membership card, which you can use for proving your identity or the right to access the requested information from the Internet [30].

The functionality of digital certificates is based on the coding technology using generated private key and public key. The private key is kept by the owner of the certificate, but public key is distributed to the persons with whom the relations are maintained. The message is encrypted using a private key, decrypted using only the public key which is related to the private key. If the message being sent is encrypted with a public key, it can only be decrypted with the private key. The institution providing the certification service creates and issues a digital certificate by signing with its own private key [30]. The use of the digital certificate provided an opportunity to verify the authenticity of the user's rights to a specific key and in this way to prevent the unauthorized use of the private key.

Use of electronic signature. To connect to various electronic banking systems or use electronic government services, one of the most common authentication methods are the use of an electronic signature:
i) any data that is linked with other data can be annotated to identify the person signature and confirm the data authenticity;
; ii) set of tools (lar related, mathematical, etc.), which together are necessary for electronic signature. It should be noted that a digital record is just a mathematically based sequence of data coding, which, due to its unique characteristics, is used to perform basic functions - authentication and establishing originality.

The requirements for secure electronic signature are related to the requirements of the Civil Code for the security of the authorized text and the identification of the person signature. The following requirements are necessary for secure electronic signature:
a) is clearly related to the signaturing person;
b) allows to identify the signaturing person;
c) it is created by tools/apps that the signaturing person can handle according his own needs only;
d) it is strongly related to signatured data, so any alteration of this data is surveillance.

The use of electronic documents has the same legal force as written documents. The use of such an electronic instrument is carried out by means of the dedicated special tools.
1. USB storage device. When choosing this electronic method of signing, the user must install the software for the USB storage device on his computer, and the use of the cryptographic USB storage device requires a USB connection on the computer.
2. Use of the mobile app. After signing an agreement with the certain company, the mobile service provided by this company allows to use a local SIM card with a secure cryptographic module installed. Therefore, it is possible to obtain a PIN code for the digital use of the certificate, which allows connecting to various online systems and electronically determining the identity of the person signing it.

3. Use of a personal identity card. Since 2009, citizens of Lithuania have been able to use chip-based personal identification cards. Identity cards can be used as a tool for creating electronic signature. The person registry services have created digital certificates that confirm electronic documents created using personal identification cards.

Use of electronic timestamp. A timestamp is a sequence of characters identifying when a certain event occurred, usually giving date and time of day, sometimes accurate to a small fraction of a second [31]. Timestamp can be defined as the data that are logically related with other data and confirms the existence of those other data until the time specified in the timestamp. The time management service provides the regulations and public availability of the time management service by publishing the regulations on its website. Without having its own website service provider, organiser must ensure the free and public usage of the service. The regulations must specify the entire procedure for the creation and management of the timestamp, the identifier of the timestamp rules, all the necessary information related to the provision of the timestamp formation services.

Timestamp can be used to sign [32]:
a) contract in electronic form, for banks, insurance companies or others;
b) electronic documents to preserve their value as evidence;
c) statements and letters sent electronically to public administration institutions;
d) internal electronic documents to protect them from forgery and backdating;
e) system logs, in order to protect them from forgery;
f) electronic invoice sent to recipients in electronic form;
g) electronic documents stored on a personal computer in order to protect them from forgery and backdating;
h) computer programs to protect them from counterfeiting and viruses.

2. Organization of Electronic Elections

2.1. Principles of Organization

The right to vote belongs to the basic principles of democracy. All principles of democratic elections and referenda must be maintained even after the establishment of the electronic election system. Electronic elections must be so safe and reliable that they are traditional without usage of modern electronic equipment. For the electronic voting system to be attractive to the public and the traditional voting system to be implemented in the future, following essential and undoubtedly important skills must be realized [33].

Publicity and global development. It is possible to ensure that all voters with the right to vote can participate in the elections, and voter identification and registration can be carried out by legal tools. For realization, the most important rules must be implemented as presented in Ref. [34]:
a) every person who has the right to express his opinion can do so;
b) the possibility to participate in elections must be guaranteed by law;
c) voting technologies and equipment must be clearly explained to voters and there must not be any restrictions on dealing with the technologies used in the election process;
d) electronic voting is only a secondary tool in the context of basic voting;
e) the infrastructure used for voting must be accessible to all voters.

The guarantee of freedom of choice. This principle ensures that the voter was not forced to use the electronic voting system, that is, he was not technologically challenged to express his opinion. Essentially, several other aspects must be considered when ensuring this freedom: the voting system must provide the voter with the opportunity to vote with a "blank ballot".

Principles of equality. The equality of the candidates and candidates participating in the elections must be ensured, that is, the equality of the rights of the voters who have the right to vote. The main e-voting requirement could be formulated as follows: paper and electronic ballots must be equivalent. For implementing this principle, it is necessary to ensure the same possibility for political parties to monitor the e-voting system and the e-voting process. According to the advice of some experts, e-voting must be realized earlier than "ballot-elections" (voting using bulletins).

Privacy Policy. Due to this principle: a) e-votes will be hidden all the time until the final counting;
b) no possibility to relate any person who voted and his vote;
c) the phases of registration for voting and voting will be clearly separated;
d) any user will not be able to provide information about his choice by any means related to the voting system.

In the voting system, the possibility of accurate counting and recounting of votes must be ensured, without identifying the person who voted [34].

The principle of directness. This principle determines the behaviour of elections: each vote must be directly recorded and counted. In order to simplify the e-voting systems and to track their operation, most often all votes received during the election period must be stored in encoded form and must be decoded only after the election.

Democracy principle. According to this principle the compliance of the voting systems with the usual traditional voting systems must ensure. There are certain essential requirements that must be met in the e-voting system. These requirements refer to the legality, transparency, security, and accuracy of e-voting systems being developed or created. Users of the e-voting system must understand how the system works, but sometimes it is impossible. Some persons do not have the basic knowledge required to use information technology. In other words, trust in e-voting systems is based on trust in technology and the willingness of the voting person to acquire and use technology.

2.2. Models of Electronic Elections

Two models of electronic elections could be realized: single-phase model, and n-phase model.

Single-phase model. Certain voter who wants to express his choice in the election can do in one simple action. Voting takes place in one cycle. In such model, voter does not need to identify himself when voting. This model is difficult to apply to electronic elections provided remotely, as there is no way to ensure that the user will not vote twice. The model of such e-voting system can only be used for e-voting provided in polling stations, since there it is possible to control who is preparing to vote.

The availability of this system for regular voting is only such that the voter, coming to the polling station, cannot damage the ballot paper and in any case must express his choice. This model can be applied together with the two-phase model, and in the future, it could replace regular voting in polling stations, as it would be better to use electronic ballot boxes than paper ballots [36]. Under the conditions of modern democracy, the one-phase model would simplify and speed up the process of calculating election results, reduce costs. Otherwise, one-phase model would not give voters the oppor-

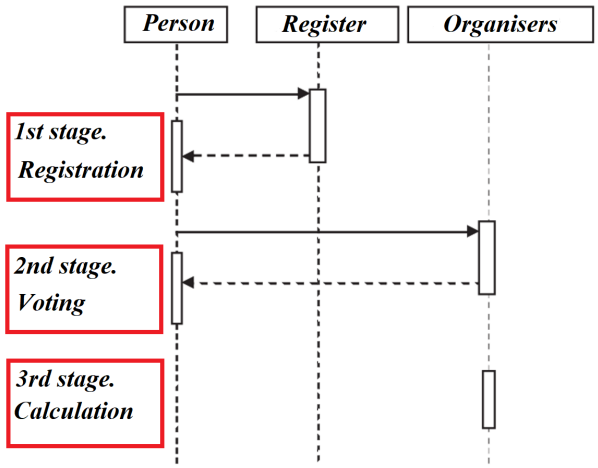


Fig. 1. N-phase model of electronic elections.
Adapted according to Ref. [36].

tunity to vote in requested geographical area, only in the place where they were at that time.

Two-phase and n-phase model. Most electronic voting systems are based on a two-phase model as presented in Fig. 1. In the first phase of e-voting, the voter must identify himself in the e-voting system, which gives him a certain permission to vote. In this way, the voter gets access to the second and main phase of the model - voting.

Sometimes the electronic election system may require additional actions from the voter during the initial (registration) and secondary (primary voting) phases. Using so-called n-phase model, e-voting person receives the messages that require the person to identify himself in separate steps in several identification systems of responsible institutions. In such a situation, the registration phase is divided into several phases and such an electronic voting system is based on the n-phase model.

2.3. Implementation of Electronic Elections

Initially it is important to consider the risk factors. The list of general requirements for electronic information security [37] defines the most important risk factors that can have an impact on electronic information security:
a) inadvertent error, subjective error (errors in electronic information management, erasing of electronic information, incorrect presenting of electronic information, physical power failures of electronic information, failures of electronic information distribution over networks, software errors, incorrect operation, etc.);
b) intentional error (unauthorized use of the information system to hack an electronic information, destruction of electronic information, malfunctions for data distribution over networks, security holes, etc.); c) *force majeure*.

Analysis of the voter's device. One of the most insecure channels of Internet voting is the voter's computer, mobile phone or other mobile devices. Personal computers are poorly monitored and poorly protected against malware attacks. When it comes to remote voting, the computers that are used to record and process votes are outside the control of the institutional supervising the votes. Due to that, the election supervisors cannot do much about it for increasing the voters' attention to these problems [38].

Hackers are constantly scanning millions of computers looking for the easiest to invade. Jefferson et al [39] observes that "computers in Internet cafes and public libraries are still more insecure. Spyware and other programs can be installed in them". When voting at the workplace, voters' risk that wheel. Internet systems and browsers are not protected from malicious programs, which can be downloaded by users or other persons using this computer due to inadvertence. A malicious program installed or downloaded into voter's computer or mobile device without the voter's knowledge can hack the vote cast by the voter or can record the fact of voting. To check information where the e-ballot was fixed, voters could observe the inconsistencies if they had access to a trust computer. This situation is problematic, because allowing voters to confirm their votes could show the realized choice of e-voting.

If the falsification occurs during the sending of the e-ballot, election officials may have no way of distinguishing between the inconsistency of the e-ballot and user error" [38]. Therefore, the possibility cannot be ruled out that the remote voting platform is being used, voters can try to manipulate it: to present more than one vote, having the content of the vote verified, to sell it. That is, trying to exercise their rights to damage the voting system, change the election results, or change the reliability of the election results.

Problems of election security. When analysing the problem of election security, big attention is paid to external threats. This is not correct approach that should be followed, because it is easier for hackers to access the system from the inside. Three main groups of potential internal threats are distinguished [36].
1. Legal users of electronic voting systems. They can search for exploits or security issues in the electronic election system and, having enough technical knowledge and a sufficient list of interested persons provided to them, can use the electronic voting system. These actions are performed most often for financial gain.
2. Persons who may seek to use the privilege of administrators of electronic voting systems to use the functionality of electronic voting systems. Representatives of this group most often seek to use the employees of other organizations that are developing the electronic voting systems. Such employees must have sufficient knowledge and understanding of electronic voting systems. The main motivation of the perpetrators of these threats is financial gain or simply personal satisfaction, self-realization by performing illegal activities. Administrators of electronic voting systems of any level must be morally prepared for such actions by hackers (outside parties) if they are provided with all the information about security "holes". ".
3. Civil servants who have access to electronic voting systems but are not related to the acquisition of electronic voting systems. These persons can participate or lead the internal "attackers" of the electronic voting system. The reasons for which these persons may commit illegal acts are most often financial or simply unspecified personal goals.

The five main groups of potential external threats are presented in Ref. [40].
1. Alone hackers looking for possibilities to disrupt electronic voting systems just to get personal satisfaction by attacking or hacking the state system. In this way, the attack is a protest against government policy. These hackers are most often looking for possibility to access the data, damage it, or use it for personal gain, for example, in many cases, to sell for illegal disclosure.
2. Another group of hackers is different from alone hackers: criminal organizations or individual criminals. These groups, for example, information brokers, may want to have unauthorized access to electronic voting systems to use the resources of these systems for personal purposes.
3. Groups of protesting persons or so-called hacktivists may try to

influence their actions in relation to electronic voting systems with the aim of showing seriousness in using these systems for voting purposes. Main purpose is to damage these systems or to obtain data for personal purposes or damage the information contained in the voting system.
4. Foreign intelligence services may be interested in receiving information about a person. In the future, they could use this information for surveillance. These services could carry out the formation of the country's politics or manipulate the available voting information to influence the voting results.
5. Terrorist organizations may be interested in information about private individuals stored in electronic voting systems. These organizations, using the available knowledge, may be interested in, for example, organizing terrorist acts. They can use electronic voting systems and the information collected during voting to find out what the voting perspectives are. This way allows to influence the voting results and to obstruct the smooth voting process.

3. Implementation of E-voting

Internet voting (I-voting) is one of the types of electronic voting (E-voting). This means that person's voice is reproduced and/or calculated by a computer system. Necessary condition for I-voting must be satified: during online voting, voters cast their votes online using their personal devices. According to Valadkevičius [6], online voting faces two unique and complex challenges.
1. Election results often fundamentally determine the statehood for several years to the year, so this is an extremely important result. Due to this determination, there are always strong interests to destroy this process for the benefit of certain interests. Therefore, when testing election innovations, it is necessary to ensure comprehensive and reliable security systems that protect elections from possible manipulation.
2. Voters' scepticism of election innovations. The main mission of democratic elections is to ensure public trust that the government really represents the nation. It is the hope that the government elections will be conducted fairly, which creates the legitimacy of democratic government. However, innovations in the field of elections often require caution: transparency and explanations are required. Ignorance makes it easy to mistrust.

3.1. Benefits of Internet Voting

1. Voter turnout. Although many critics of Internet voting claim that there is no evidence that Internet voting increases significantly voter turnout, any increasing is evident. If one third of all voter's take part in election voted online, a 2.5-4% increase in election activity can be expected.
2. Validity of voting. I-voting would be extremely useful for Lithuanians abroad, living in the far corners of the world: in many Asian countries, in Latin America. I-voting would ensure that these voters could cast their vote in the elections of Lithuania Republic.
3. Accurate and unbiased counting of votes. When I-voting is implemented, the process of casting and counting votes would be centralized, monitored by competent observers, and pre-agreed rules would be made public. Voting results could be damaged only if outside hackers break into the system.
4. Ensure safe and reliable voting. The applied full verification system means that it is possible to verify and reliably make sure that each vote of any voter was accepted, credited and included according to following routines: i) cast as intended, ii) recorded as cast, iii) counted as recorded. These three steps formed the chain of the "journey" of the I-voice, and after proving the integrity of each chain, the integrity of the entire network is proven.
5. Increased level of vote encryption. I-voting could increase the security, help with voting at home and abroad. By voting online from home, disabled and elderly people will not have the physical contact to carriers of election ballots, and citizens from abroad will be not able to vote repeatedly. So, these threats will be negligible.

3.2. Risks of Internet Voting

It is considered that online security arguments are one of the main ones on which Internet voting is criticized. Below are the risks of online voting and possible ways to resolve them. This information is based on the analyses of the European Council, OSCE/OHIDR, IFES and other international organizations, including the good practices of Norway, Estonia, Switzerland, and Canada [41].
1. Falsification of votes. The I-vote, realized by I-voter, could be damaged or changed without the voter knowing. After counting procedure, election results are different than the voters intended. This risk could be analysed for three stages of online voting: casting, storing and counting. For casting, voter's computer is responsible for data transfer to the server. This risk can be eliminated by using the vote verification function, which allows the voter to make sure, using the remote device, that the vote went to the counting server without being hacked. For storing, vote storage server is responsible. This risk can be eliminated by keeping the cast vote data on several different remote servers, which would be synchronized. For counting, risk of errors can therefore be eliminated by copying data to several servers (mirror service). The second solution is to give the possibility to any voter to download all decrypted election data (without authentication of any voter) from the vote server.
2. Decryption of votes as a technical cut. The main risk of counting votes is that online voting data can be hacked, and voter information recorded in the ballot can be revealed. In legal acts, it is necessary to provide the protocols and standards for clear data transfer, processing and calculation.
3. Revealing the voting secret as a social cut. One of the main social risks associated with online voting is the purchase of votes and the announcement of votes in a certain social network. Therefore, the risk remains that the voter may be pressured or forced to vote after the end of the online voting period.
4. Lack of qualifications of election implementers. It is recommended that online voting projects adopt the standards that have been tested and publicly updated. When preparing the parameters and requirements of the future system for the suppliers and the institutions acquiring the system, it is necessary to be guided by the analyses and reports prepared by the relevant international organizations.
5. Public distrust and quality control. There are two most important factors that lead to public distrust: a) absence and (or) of a high-quality and secure online voting system; b) lack of appropriate information about this system. To solve first problem, several technical and operational security and quality assurance systems are required. To eliminate the second problem, it is necessary to ensure that essential information about the operation of the system is publicly available and accessible to the public. The proposed arrangements fully reveal the system's operating principles and mechanisms: a) detailed audit for system checking in detail, and the conclusion will be present to the public; b) open source software and open protocols allow all programmers to analyse source code and to estimate the security bugs of the system; c) election monitoring

must be organized, observers during the counting of votes would ensure that everything is carried out in the prescribed order.

Summarizing the possible threats, it can be pointed out that online voting is a high-risk project due to two reasons [41].
1. Due to the importance of certain elections for the statehood, there are strong interests to redirect the result for the benefit of individual interested groups.
2. Basic mission of democratic elections is to ensure the public's trust that the government elections were conducted fairly, which creates the legitimacy of democratic government. Innovations in the field of elections can easily cause disappointment among potential voters, so exceptional transparency is often required. When election innovations are implemented, it is necessary to ensure comprehensive and reliable security tools that protect against possible manipulation of elections and distortion of results. Only such tools can ensure high standard of transparency and publicity.

Proponents of online voting therefore hope to increase voter turnout, while critics fear that due to the current cyber security situation, the online voting system may be an easy target for attacks. Finally, online voting system may be aimed at to change election results. Additional problems about ensuring of anonymity are not yet solved.

4. Systems of Electronic Election

Various electronic voting systems have been developed and tested in many European countries. Outside of Europe, the USA and Brazil are currently engaged in electronic voting, while Mexico and Central and Latin American countries are considering it. According to the setting of clear goals of electronic voting, the countries of the world can be divided into three groups.
1. Israel, together with Scandinavian countries. These countries rely on the traditional elections using paper ballot. There are no political plans that include electronic voting.
2. Countries that have introduced or are introducing electronic voting machines, which do not have any plans to introduce remote electronic voting throughout the country (for example, the USA, Brazil, Russia). 3. Sweden, Canada, France, and Estonia. These countries are planning to introduce or have already introduced remote electronic voting throughout the country.

One of the first large-scale electronic voting systems was implemented in USA. Secure Electronic Registration and Voting System (SERVE), acquired by the US Department of Homeland Security department was planned to be used in 2004 for US voters abroad. However, Pentagon refused to use this system due to serious security threats. The security analysis [50] done by independent researchers has caused a lot of discussion, and conclusions state that since the risk of large-scale and successful attacks of hackers is so high, it is recommended to stop the development of SERVE. Moreover, similar systems should not be attempted in the future until the infrastructure of the Internet and personal computers has been fundamentally redesigned and improved, or until a fundamental security arrangement has been implemented.

4.1. Electronic Elections in Estonia

Many countries use electronic voting machines at polling stations, but Estonia did not choose this way. Electronic voting in the context of Estonia means remote voting over the Internet. The most important goal is to provide voters with the opportunity to vote and thereby increasing voter participation.

Internet voting system. Until now, Estonia is the only country that has introduced online voting in elections on a national scale. Internet voting was used in Estonia in the municipal elections (July 2005) and in parliamentary elections (2007). In January 2005 electronic voting system was tested In Tallinn, during public testing before the municipal elections. Also, local referendum about the location of the Freedom Monument was conducted using this system. In 2002 political agreement on the implementing of electronic voting in Estonia was concluded.

Legal framework of e-voting was started in 2002 after adoption of the Local Government Election Law, which allows voters to vote online on the website of the Central Election Commission during early voting.

In 2005, after the main election commission approved the online voting system, the law on the registration of the municipal election law was adopted, which formulates the voter's right to adjust his e-vote: a) by voting again on the Internet or b) at the polling station on the day of the election. This law is related to realization of the voting freedom.

The law was vetoed by the President Arnold Rüütel, arguing that the law created non-equivalent conditions for electronic and traditional voters, since online voters are allowed to change their votes, others do not have such an opportunity. After Parliament rejected the president's veto, he appealed to the Supreme Court regarding the constitutionality of the law. The Supreme Court rejected the resident's complaint, explaining that the law does not violate the Constitution and does not violate the equal rights of voters, as all voters have the right to vote online. The laws were finally approved only in September 2005. In 2005 9,317 voters voted online in the municipal elections in Estonia. It was 0.9% of registered voters or 2% of voters who participated in the elections.

During the election, there were no serious technical problems, apart from the fact that on the third day of voting, the electronic voting was disrupted for one hour due to the problems of certification service provided by *Sertifitseerimiskeskus AS*. No attacks that could cause system security problems were recorded. The auditors confirmed that the electronic voting system worked correctly, there were no malfunctions or problems that could raise suspicions about the correctness of voting and the trustworthiness of the system. In April 2007 30,275 voters voted online in the parliamentary elections [19].

Identification of voters. Online voting in Estonia is growing using an electronic identification (ID) card. In February 2006 the number of ID card holders was 65%, July 2007. - about 80% of adult recipients [43]. Electronic identification card is valid for all Estonian recipients since January 1, 2002. It is used for remote personal identification and electronic signature. Although there are currently 1 million electronic ID cards only 70,000 users are active users. This explains the relatively small number of people who voted online. In addition, those who wanted to vote on the Internet must be equipped by computer with Internet access, an ID card with a certified certificate and PIN code, a card reader and serial software.

Fig. 2 represents online voting system in Estonia [44] which is based on the "two-envelope" system, currently with a ballot. The voter's vote is encoded when sending it to the voting server. Encoded vote can be kept in an anonymous inner envelope, as an analogy with written voting. Therefore, the voter signs in with an electronic signature. This means that person identification data is presented in the inner envelope. Then, before the counting the votes, the encoded vote and the digital record with personal data are separated. Electronic votes are then decoded and counted.

Internet voting in Estonia has been recognized as successful. The

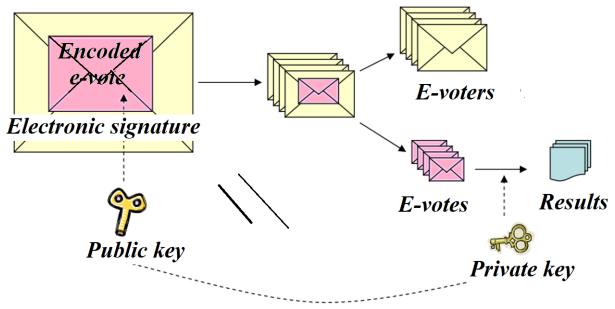


Fig. 2. Online voting system in Estonia.
Adapted according to Ref. [44].

OSCE/ODIHR report [45] states that the administration of the elections was transparent. It was stated that the Central Election Commission of Estonia tried to minimize the risk and ensure safety. However, the report notes that the scope of testing and auditing could have been greater. There is a discussion about possible problems when the number of online voters increases. The report contains recommendations on how to increase the security and reliability of online voting. The conclusions of the report state that "as long as the problematic aspects of online voting will not be effectively resolved, the government should seriously reconsider this question: should online voting be considered as a voting method, or should it be used to a limited extent or not at all? Or may be refused?".

If such an election scheme would be transferred to Lithuania it would be an important and positive arrangement. Usage of the online election system in Lithuania will automatically avoid threats related to fears about the secrecy of voting. Several opposite opinions exist: voters who do not use the online election system will be disadvantaged. When a situation arises, voters will obtain different rights: those who vote online will be able to change their choice, while those who only vote at the polling station will no longer be able to do so. This could be considered a weighty argument for politicians. However, let's not forget that "online elections will be relevant for each voter and only the choice of each voter, whether to use technology or not, determines this emergence of advantage [21].

Principles of voting on the Internet. In Estonia, Riigikogu election law intended for definition of online voting [46]. One of the main features of online voting is that the voter votes himself, and in the cases provided by law, the voter can change his vote.

Time interval. Voting can be done online for 7 days - from the 10th to the 4th until the election day. The possibility to correct the voice on the Internet is realized for all e-voters. During the online voting period, the voter can change his vote online - in this case, the last online vote is counted.

The supremacy of ballot voting. What is happens if the person previously voted online during the early voting period and then decided to go to the polling station and votes again using the regional ballot paper? According to e-voting rules, the vote cast online is invalid. Therefore, a voter can no longer correct (change) his vote online or using a paper ballot. On election day, votes cannot be changed online.

Similarities between online voting and traditional voting. I-voting is based on the provisions of election laws and basic election rules. Thus, voting is uniform and secret, only voters can vote, each person can cast only one vote. It must be impossible to check how the voter voted. Voting is safe, reliable, and proof. Voters should be able to cast their vote freely and without external pressure or influence.

Invitation to vote online for gifts to voters or other influence is prohibited. Also, collective online voting events are prohibited (Internet voting services, information centres, etc.). Such activities can be considered a violation of electoral freedom. The voter votes independently on the Internet. The use of mobile phone card belonging to another person for voting and corresponding transfer of card passwords (PIN codes) to another person are prohibited. To avoid security risks, only a known computer is used - either owned by the voter or a person whom the voter trusts.

4.2. Electronic Elections in Switzerland

Development of Internet voting. In 2000, Swiss federal government has decided to adopt an Internet voting project, which it supported with the argument that after adopting electronic voting, Switzerland could play the role of a leader and to demonstrate to other countries that the science of information technologies can serve in the implementation of the elements of direct democracy [57].

The main reasons for the adoption of online voting systems in Switzerland were related to the increase in the success of elections due to the significantly increasing number of elections and referenda, decreasing voter turnout, successful voting through the postal service and the positive influence of this voting method on voter turnout. The implementation of online voting was also determined by such factors as the high level of Internet penetration. 10% of Swiss citizens are abroad, and only 2% of them register to vote. In addition, the low level of voter turnout for people younger than 40 was recorded. Generally, public opinion data showed a high level of idea popularity to vote using Internet [47].

In 2000, Swiss Confederation has developed a centralized government structure that allowed it to maintain control over the Internet voting project. Federal government signed an agreement with three cantons (Geneva, Zurich and Neuchâtel), which voluntarily decided to acquire the Internet voting project. The federal government agreed to finance 80% of project costs until 2005, and the cantons agreed to slow down the Internet voting system and make it available free of charge to any of the remaining 23 cantons. Thus, the Swiss confederation would retain the right to control the project for an indefinite period.

In October 2011, 22 thousand Swiss expatriates had the opportunity to vote online in the recent federal elections. After this elections Swiss Confederation decided to grant the right to vote online to all voters living abroad. Priority and opportunity to vote online was guaranteed for voters from abroad are given [48].

Other arguments for the legalization of Internet voting are related to the improvement of vote counting systems, the immediate inspection of damaged ballots, and the acceptability of such a voting method to some voters.

The right to vote online. Since February 2014 all Swiss citizens living abroad have the right to vote online. The only requirement is to register for voting in those cantons that provides online voting. Such a right is guaranteed to voters from states belonging to the European Union and countries that have signed the Wassenaar Agreement. States ratifying the Wassenaar Agreement agree not to punish individuals for the use and use of both (civilian and military) devices and technologies. Cryptographic programs partly fall into the category of these two distinct tools and technologies. Thus, the Swiss government wants to protect its citizens from possible legal responsibility for the use of such programs in countries that have not signed the Wassenaar Agreement, thus limiting the possibility of on the Internet. The federal law allows foreign voters to vote in federal elections and referenda. The canton of Geneva granted such

voters role rights in the cantonal legislature. For foreign voters to be included in the voter lists, they must register their place of residence at the Swiss consular office and renew this registration every four years [48].

Some voters living in Switzerland also can vote online in the cantons of Geneva and Neuchâtel. Other cantons implementing pilot projects for online voting are targeting voters from abroad. The long-term goal is to provide the possibility of online voting to all Swiss voters in the future. The Swiss Confederation (federal government) has set limits for each canton using Internet voting - no more than 30%. Voters in every canton that uses Internet voting can vote online. To avoid the above-mentioned arbitrariness, the government of the canton of Geneva has selected 15 municipalities, in which there is about a third of the voters of the entire canton. In these municipalities, all citizens can vote online without prior registration. Swiss voters abroad are not included in the 30% of voters. In addition, 30% of voter's registration does not apply to cantonal and self-governing elections or referenda. In 2020, 13 cantons offer online voting to voters: the cantons of Geneva, Neuchâtel and Zurich use their own online voting systems, and the remaining 10 cantons use one of the following the online voting system of these three cantons [48].

Internet voting system. After completing the so-called electronic ballot box removal procedure, the Geneva canton's online voting site is now open to the public. Then, the voter uses the Internet browser to connect to the Internet site [49] and edit the voting procedure. To activate the voting procedure, the voter enters his voting card number from identification field on the website and consults with the current legal information (prominent response for electoral laws, etc). In the next step, the voter fills in the electronic ballot, verifies the choices made in the ballot and enters secret data (password, date of birth and municipality of origin) in the authentication form. After completing these actions, the voter will be redirected to the confirmation page, which provides confirmation information about the successful saving of the cast vote in the electronic ballot box [48].

A single use voting card, necessary for every online vote, is sent to the voter by post. The voting card contains a unique voting card number that identifies the voter in the election system, regardless of the way he votes. The voting card number is used to identify the user to confirm his connection to the voting network. Voting card number represents the confidential information, and all transfers in any form must be provided in encrypted format [48].

Online voting ends at noon on Saturday, i.e. u. one day before the election day when voting starts at the polling station. The possibility of revoting via the Internet voting system of Geneva does not provide for the possibility of revoting via the Internet and other ways (in writing or on election day at the polling station). In the Geneva electoral system, there is unique list of voters for all three voting methods (online, through post-office and in the polling station), and due to that, no possibility to vote more than once.

In I-voting, voters are required to enter their municipality of origin and date of birth. These data are not available in public registers. The identity of origin is also indicated on the identity card and passport of a Swiss citizen. Also, the election organizing service (Service des votations et elections, SVE) organizes a telephone survey for 4000-8000 of voters who voted to make sure that the voters voted freely [48]. In case of voters activity 50%, the volume of respondents is 4-8%) .

In summary, the Internet voting system in Switzerland due to the voter identification mechanism would not be suitable for the case of Lithuania, as it would facilitate the possibility of reducing the speed of voting and voting for other persons. It would only be enough to know the date of birth of other voters, the municipality of origin and verify the voter card of another person. In addition, in Switzerland, there is no possibility to re-vote either online or in a traditional way.

Conclusions

The theoretical analysis of the development and application of cyber security management tools for electronic elections showed following statements.

1. Electronic elections shall be defined as any type of voting in which electronic means are involved. In the context of e-voting, it is important to distinguish between voting using electronic machines and electronic remote voting: Internet voting and electronic voting.
2. Any electronic voting system consists of six basic elements, which are known as traditional voting systems: voter registration, authentication, voting, vote control, calculation of events, audit.
3. For electronic voting systems to be attractive, they must acquire the following essential and undoubtedly important characteristics: universality and universal participation, freedom of choice, love, secrecy, immediacy, democracy. Models of electronic elections can be: one-phase model, two-phase and n-phase model.
4. Main causes of potential internal threats represent users of e-voting systems: legal users as e-voters, administrators of e-voting systems, civil servants who have access to e-voting systems.
5. The following groups of potential external threats are distinguished: individual hackers, criminal organizations, groups of protesting persons, foreign intelligence services, terrorist organizations.

Case analysis of the application of cyber security management tools and technologies in electronic elections showed following outcomes. Due to the best benefits of electronic voting, three different groups of countries can be distinguished:

a) main group of countries rely on the traditional "ballot" voting system, without any politicians considering electronic voting;

b) countries which has introduced electronic voting machines, but they do not have any plans to introduce remote electronic voting throughout the country;

c) Sweden, Canada, France, and Estonia have already introduced remote electronic voting throughout the country.

Lithuania has good opportunities to support Internet election research from other countries, to carry out continuous public education, to apply the latest security measures. There is a not negligible but slight threat, that public distrust of the state and its institutions may increase in case of cyber attack disrupts electronic voting system and damages the choices of voters.

Abbreviations

ATM	-	Automated Teller Machine
EMISARI	-	Emergency Management Information System
	-	and Reference Index
HAVA	-	Help America Vote Act
IFES	-	International Foundation for Electoral Systems
NJIT	-	New Jersey Institute of Technology
OSCE	-	Organization for Security and Co-operation in Europe
OHIDR	-	Office for Democratic Institutions and Human Rights
SERVE	-	Secure Electronic Registration and Voting System
SVE	-	Service des Votations et Elections

References

1. 2016 m. spalio 9 d. Lietuvos Respublikos Seimo rinkimai (in lith.) - https://www.vrk.lt/2016-seimo.
2. Galdikienė, L. () Kai pučia vėjas, reikia statyti malūnus (in lith.) - https://www.lb.lt/lt/naujienos/l-galdikiene-kai-pucia-vejas-reikia-statyti-malunus
3. Radžiūnas, V. () Kodėl pusė Lietuvos rinkėjų nenori dalyvauti rinkimuose? - https://m.klaipeda.diena.lt/naujienos/lietuva/politika/ kodel-puse-lietuvos-rinkeju-nenori-dalyvauti-rinkimuose-671568
4. Discusion: online voting is a long process that can start with voters abroad - https://www.delfi.lt/en/world-lithuanians/ discusion-online-voting-is-a-long-process-that-can-start-with-voters-abroad-84663155.
5. Agafonov, K. (2016) Apie elektroninį balsavimą paprastai (in lith.) - http://apzvalga.eu/apie-elektronini-balsavima-paprastai.html.
6. Valatkevičius, S. (2015) https://old.kurklt.lt/wp-content/uploads/2015/10/Balsavimo-internetu- %C4%AFgyvendinimo-ir-elektronini%C5%B3-apylinki%C5%B3-projektas.pdf
7. Kosowska, E. (2011) Frost & Sullivan, + https://www.newswiretoday.com/news/91994.
8. Štitilis, D. (2013) Kibernetinio saugumo teisinis reguliavimas: kibernetinio saugumo strategijos (in lith.)- *Socialinės technologijos* 3(2013)189-207.
9. Information Technology Industry Council (ITI) - https://www.itic.org/about/.
10. EU Cybersecurity Strategy - https://digital-strategy.ec.europa.eu/en/policies/cybersecurity-strategy.
11. Warren, M.; Štitilis, D.; Laurinaitis, M. (2023) Cyber Lessons that the World Can Learn from Lithuania - In: European Conference on Cyber Warfare and Security (2023) 517-524
12. Lietuvos Respublikos kibernetinio saugumo įstatymas - https://e-seimas.lrs.lt/portal/legalAct/lt/TAD/f6958c2085dd11e495dc9901227533ee.
13. Kiškis, M.; Petrauskas, R.; Rotomskis, I.; Štitilis, D. (2006) Teisės informatika ir informatikos teisė (in lith.) Textbook. - ISBN 9955-19-048-5 - Vilnius, Mykolo Romerio universitetas.
14. Jastiuginas, S. (2011) Informacijos saugumo valdymas Lietuvos viešajame sektoriuje (in lith.)- *Information & Media* 57(2011) 7-25 - https://doi.org/10.15388/Im.2011.0.3137
15. Council Directive 2008/114/EC of 8 December 2008 on the identification and designation of European critical infrastructures and the assessment of the need to improve their protection - https://eur-lex.europa.eu/legal-content/en/TXT/?uri=CELEX:32008L0114
16. Kiškis, M.; Kraujalytė, A. (2005) Informacinių technologijų įtaka teisiniams-politiniams procesams e-valdžios kontekste (in lith.) - *INVENT* (2005) 771-778.
17. Recommendation Rec(2004)11 of the Committee of Ministers to member states on legal, operational and technical standards for e-voting. Council of Europe, Committe of Ministers. - https://www.coe.int/t/dgap/goodgovernance/Activities/Key-Texts/ Recommendations/00Rec(2004)11_rec_adopted_-en.asp.
18. Enguehard, C.; Graton, J. (2008) Electronic Voting: the Devil is in the Details - https://hal.science/hal-00274635.
19. Ramonaitė, A.; Barcevičius, E.; Jurgelevičiūtė, D.; Medelis, Z.; Juozapavičius, M.; Jokubaitis, A.; Lipskis, D.; Kirvelytė, L. (2008) Elektroninio balsavimo galimybių studija - http://www.lnes.tspmi.vu.lt/wp-content/uploads/2021/12/Elektroninio-balsavimo-galimybiu-studija-6.pdf.
20. Janonis, T. (2012) Kuo skiriasi el. balsavimas nuo balsavimo internetu? - https://www.delfi.lt/archive/kuo-skiriasi-el-balsavimas-nuo-balsavimo-internetu.d?id=59528505.
21. Renner, R.L., Bechtold, R.M., Clark, C.W., Marbray, D.O., Wynn, R.L., Goldstein, N.H. (1974) EMISARI: A Management Information System Designed to Aid and Involve People. - In: Tou, J.T. (eds) Information Systems. - Springer, Boston, MA. - https://doi.org/10.1007/978-1-4684-2694-6_-13
22. Rupp, C. (2004) E-Democracy in E-Austria - - https://subs.emis.de/LNI/Proceedings/Proceedings47/Proceeding.GI.47-2.pdf.
23. HAVA, Help America Vote Act, US Election Assistance Commision (2017) - https://www.eac.gov/about/help_america_vote_act.aspx.
24. Visvalingam, K.; Chandrasekaran, R. (2021) Secured Electronic Voting Protocol Using Biometric Authentication - *Advances in Internet of Things* 1 (2021) 38-50 - doi: 10.4236/ait.2011.12006.
25. Electronic voting in Switzerland - https://en.wikipedia.org/wiki/Electronic_voting_in_Switzerland.
26. Cabello Pardos, A.B.; Hernández Encinas, A.; Hoya White, S.; Martín del Rey, A.; Rodríguez Sánchez, G.(2007) A Simple Protocol for Yes-No Electronic Voting - *IJCSNS International Journal of Computer Science and Network Security* 7(2007)72-76.
27. Bogdan, J.; Veselaja, O. (2011) Analiz sushchestvujushchich sistem golosovanija (in rus.) - *Eastern-European Journal of Enterprise Technologies* 1 (2011)33-49.
28. Commission Implementing Regulation (EU) No 310/2014 - https://www.legislation.gov.uk/eur/2014/310/adopted.
29. Nathanael, P.; Tanenbaum, A. S. (2009) The Design of a Trustworthy Voting System - In: ACSAC '09: Proceedings of the 2009 Annual Computer Security Applications Conference - (2009) 507-517. - https://doi.org/10.1109/ACSAC.2009.54.
30. http://www.ssc.lt
31. Song, S.; Huang, R.; Cao, Y.; Wang, J. (2021) Cleaning timestamps with temporal constraints - *The VLDB Journal* 30 (2021)425-446 - doi:10.1007/s00778-020-00641-6
32. Limba, T.; Novikovienė, L. (2012) Elektroninio parašo ir laiko žymos įtaka elektroninių sutarčių apsaugai - *Socialinės technologijos* 2(2012)483-501.
33. Limba, T.; Agafonov, K. (2012) Models and Principles of Designing e-Voting Systems, Ensuring its Protection - *Socialines Technologijos* 2(2012).
34. Gritzalis, D. A. (2002) Principles and requirements for a secure e-voting system - *Computers & Security* 21(2002)539-556 - https://doi.org/10.1016/S0167-4048(02)01014-3.
35. International Working Group on Data Protection in Technology (IWGDPT) - https://www.bfdi.bund.de/EN/Fachthemen/Inhalte/Europa-Internationales/Berlin-Group.html.
36. Rössler, T. (2009) Electronic Voting Using Identity Domain Separation and Hardware Security Modules. - In: Godart, C., Gronau, N., Sharma, S., Canals, G. (eds) - Software Services for e-Business and e-Society. I3E - *IFIP Advances in Information and Communication Technology* 305(2009) - Springer, Berlin, Heidelberg. - https://doi.org/10.1007/978-3-642-04280-5_1.
37. Minimum Information Security Requirements for Systems, Applications, and Data - https://safecomputing.umich.edu/information-security-requirements.
38. Recommendations Report to the Legislative Assembly of British Columbia (2014) Independent Panel on Internet Voting - https://elections.bc.ca/docs/recommendations-report.pdf.
39. Jefferson, D.; Rubin, A.; Simons, B.; Wagner, D. (2004) A Security Analysis of the Secure Electronic Registration and Voting Experiment (SERVE) - https://www.acm.org/binaries/content/assets/public-policy/usacm/e-voting/reports-and-white-papers/serve_report_full_paper.pdf.
40. Štitilis, D.; Klišauskas, V. (2015) Aspects of cybersecurity: the case of legal regulation in Lithuania - *Journal of Security and Sustainability issues* 5(2015)47-59

41. Rubikas, I.; Valatkevičius, S. (2015) Balsavimas Internetu. Geriausios užsienio praktikos ir taikymo Lietuvoje galimybės (in lith.) - Vilnius, 2015.
42. Donauskaitė, D. (2008) Balsavimo internetu klystkeliai (in lith.) - https://www.delfi.lt/news/ringas/lit/ddonauskaite-balsavimo-internetu-klystkeliai-17525311.
43. ID - https://www.id.ee/en/rubriik/e-estonia/
44. Maaten, E. (2004) Towards remote e-voting: Estonian case - In: Proceedings of Conference: Electronic Voting in Europe - Technology, Law, Politics and Society, - Workshop of the ESF TED Programme together with GI and OCG - Schloß Hofen / Bregenz, Lake of Constance, Austria
45. Alternative voting methods and arrangements - OSCE/ODIHR report. https://www.osce.org/files/f/documents/2/a/466794.pdf.
46. Elections of the Riigikogu - https://www.riigikogu.ee/en/introduction-and-history /riigikogu-tasks-organisation-work/elections-riigikogu/.
47. Barrat i Esteve, J.; Goldsmith, B.; Turner, J. (2012) International Experience with E-voting. Norwegian E-vote Project - https://www.parliament.uk/globalassets/documents/speaker/digital-democracy/IFESIVreport.pdf
48. Juknevičiūtė, D.; Skačkauskas, M. (2015) Balsavimas internetu: užsienio valstybių patirtis ir perspektyvos Lietuvoje - https://www.vrk.lt/documents/10180/556540/Balsification+internet.pdf /a5247fe6-d96e-437d-8135-5db76da1f66f.
49. https://www.evote-ch.ch/ge

Review Article

Realizations of the Artificial Neural Network for Process Modeling. Overview of Current Implementations

Alytis Gruodis*
Department of IT, Vilnius Business College, Kalvarijų g. 129-401, LT-08221 Vilnius, Lithuania
* Corresponding author, e-mail: *alytis.gruodis@gmail.com*

Received: 01 December 2023
Accepted: 20 December 2023
Online: 30 December 2023

JEL: C45.

Abstract. This work is intended to review the most typical realizations of Artificial Neural Networks (ANNs), implemented in a Feedforward Neural Network (FNN) as well as a Recurrent Neural Network (RNN). Essential differences in ANN architecture and basic operating principles are discussed. The problems of learning processes are presented in several cuts. The advantages of prediction using ANNs have been demonstrated in several popular fields such as adaptive educology, classification of medicine and biology, industry, etc.

Keywords: Artificial Intelligence; Artificial Neural Network; Feedforward Neural Network; Recurrent Neural Network; Perceptron.
Citation: Alytis Gruodis (2023) Realizations of the Artificial Neural Network for Process Modeling. Overview of Current Implementations. – *Applied Business: Issues & Solutions* 2(2023)22–27 – ISSN 2783-6967. https://doi.org/10.57005/ab.2023.2.3

Introduction

In variuos fiels of industries, Artificial Intelligence (AI) and Machine Learning (ML) have a wide range of applications. In manufacturing to reduce downtime, and to enhance overall operational efficiency, AI could be applied for quality control, supply chain optimization, and process automation [1-2]. For development of self-driving cars and drones several complex operations such as real-time decision-making, navigation, and object detection, image recognition could be realized using AI only [3-4]. In social plane, to create communication between humans and computers, the Natural Language Processing (NLP) as subfield of AI must be implemented for speech recognition, language translation, chatbots etc [4-5]. In economic plane, performance in forecasting stock indices and future predictions could be applied using different types of networks in relation to Multilayer Perceptron (MLP) model [6].

In education, personalized learning and recomendations of educational content could be realized due to AI and ML technologies [7-8]. Possibility to create an individual learning style adapted to human represents an effective realization of one of the greatest educational issue of XX century [9].

In AI and ML, Artificial Neural Networks (ANNs) represent main realization of predictions in decision making processes. According to neuron networks in biological systems, nonlinear reswitching function of neuron was implemented in perceptrons which represent an operation unit in ANNs [10]. Behaviours presented below determine the essence of ANNs: a) learning from data and continuous improvement; b) fault tolerance; c) non-linearity; d) adaptability.

Big amount of real data is too complex for direct analysis using formula representations. After tranditional modelling, some conclusions could be questioned due to locality of solutions and impossibility to join several fields of data. Usage of ANN allows to learn the complex patterns and relationships from large weak correlated data sets of different type [11]. In many fiels of human development, amount of data increases continuously. Due to that, previously created and trained ANN can be further trained and fine-tuned to improve their prediction again and again. Basheera et al [12] accented that attractiveness of ANNs comes from their remarkable information processing characteristics pertinent mainly to high parallelism, noise tolerance, and learning capabilities. This adaptability allows them to evolve and stay relevant over time [13].

Fault and noise tolerance represents critical problem in information science, which must be solved using individual approach. For ANNs, predictions using well trained networks are not sensitive to missing of portion of data or network damaging due to network facilities. In many cases, ANNs often exhibit fault tolerance [14].

Non-linear data are obtained from big number of processes related to the real-world problems. Traditional linear models in economics cannot be sufficient for representation of such complexities. In that case, non-linear data relationships could be succesfully analysed using ANNs [15-17]. Jaiswal et al [15] claims that stock price does not follow any deterministic regulatory function, factor or circumstances rather than many considerations such as economy and finance etc.

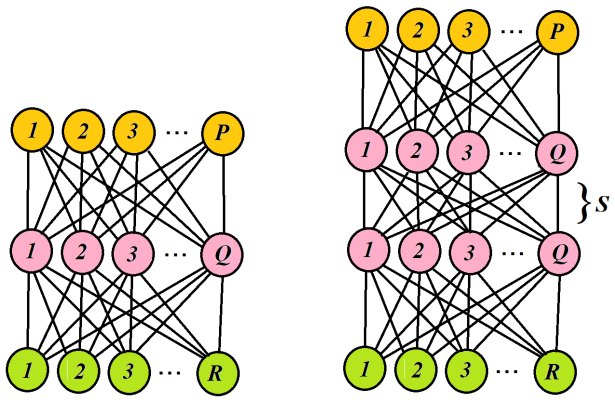
Adaptability of ANN to complex parameters is well known due to network possibility to generalize new, unknown data of different type [18]. Classifications of current changes in inputs and following predictions allow adjusting their internal parameters during the learning process. ANNs represent non-linear, data-drive, and self-adaptive methods that do not require specific assumptions about the underlying model [6]. Methods for knowledge initialization, knowledge refinement and knowledge extraction [19] represent several secial constraints of ANN imposed by nonmonotonicity (ANNN).

Generally, ANNs are necessary because learning from current weak correlated data allows us to create novel models containing complex relationships, and adapt such models to new information.

This work is devoted to review the most typical implementations of ANN, essential differences in ANN architecture to describe main operational principles, the problems in the learning processes, and fields, where predictions are welcome.

1. Artificial Neural Network: architecture

All artificial neural networks ANNs are based on the concept of neurons, connections, and transfer functions. According to Kohli [20], ANN is an attempt to simulate the brain activity. Simulated ANN as very simplified brain could be used for solving some mathematical tasks according to operational principles inherited from biological neurons in synapses. As presented in biology, real tasks such as pattern recognition or data classification could be partially of fully solved after corresponding learning process. Learning in biological systems involves adjustments or readjustments to the synaptic connections that exist between the neurons.



a) single hidden layer; b) multiple hidden layers, S>1.

Fig. 1. Feedforward neural networks (FNNs).
Hidden layer (red): number of perceptrons Q,
input layer (yellow): number of units P,
output layer (green): number of units R.

Wang [4] presented several models and algorithms of ANNs and analyses of the main ideas. Complete taxonomy of two main groups such as feed-forward neural networks (FNNs) and recurrent/feedback neural networks (RNNs) was described by Jain et al [14]. Due to the link pattern, two categories of ANNs occur: a) FNNs, where graphs have no loops; b) RNNs where loops in graphs occur because of feedback links. Gómez-Ramos et al [6] expanded previous classification up to four groups of networks used as forecasting tools: 1) feed-forward neural networks (FNNs), like the Multi-Layer Perceptron (MLP), 2) recurrent/feedback neural networks (RNNs), 3) modular networks (MN), and 4) support vector machine (SVM).

Feedforward Neural Networks. FNNs, also known as MLPs, form the foundation of many deep learning models. Layers of interconnected nodes represent an feedforward network. Term *feedforward* represents information flow in one direction from the input layer through hidden layers to the output layer:

$$input \rightarrow hidden \rightarrow output. \tag{1}$$

Fig. 1 represents schemes of FNNs in two realizations: a) single hidden layer; b) multiple hidden layers, S>1. Input layer and output layer consist of input ports, so called units, where number of units is equal to P and R respectively. Hidden layer (or layers) consist of certain number of perceptrons Q, and Q>P. This condition must be realised according to the needs of tasks and could be treated as non-mandatory condition.

Walczak et al [11] presented the problem of perceptron's quantity in hidden layer. Walczak described several empirical heuristical approaches used before in different projects.
Let us consider N_h as number of hidden nodes for an ANN, N_{in} and N_{out} are numbers of units in input and output layers respectively. Three intuitive rules for establishing the number of perceptrons in the hidden layer are presented below.

$$N_h = \frac{3}{4} \cdot N_{in} \tag{2}$$

$$N_h = \frac{1}{2} \cdot (N_{in} + N_{out}) \tag{3}$$

$$N_h = 2 \cdot N_{in} + 1 \tag{4}$$

To calculate the required number of perceptrons in the hidden layer, Bekešienė et al [21] presented several formulas that are variations of Eq.(47).

Perceptrons as artificial neurons in each layer perform a weighted sum of their inputs, followed by the application of an activation function. The weighted sum is computed using values representing the properties of link between percetrons. In hidden layer, each link has an associated value, which could be (or must be) changed during the training process. These values or weights determine the strength of influence between perceptrons.

Traditional FNNs have no memory of previous inputs and the fixed architecture. Three characteristics are important for defining the ANN [22]: a) number of hidden layers and number of nodes in each layer; b) mechanism of learning which is responsible for updating the weights of links; c) activation function used in various layers.

For FNNs, condition of stationarity assumption must be realized. FNNs assume that the underlying statistical properties of the input data do not change over time. For tasks involving non-stationary data FNNs are not suitable.

Ability to approximate any continuous function in the framework of certain conditions is one of the most important fundamental properties of FNNs, known as the *Universal Approximation Theorem* [23].

However, it is shown that the FNNs has important delimitations in network architecture: fixed input size; limited handling of time series data; sensitivity of hyperparameter.

For FNNs, a fixed number of features as input must be presented. Due to restrictions of architecture, size of input for training and for predictions must be the same. Convolutional Neural Networks (CNNs) are better suited for tasks involving image data, where the input size can vary.

For FNNs, limited handling of time series data is the serious problem. FNNs are not adapted for processing of the time series data or text item data. Long Short-Term Memory networks (LSTMs) and Recurrent Neural Networks (RNNs) are more adapted for tasks involving time series data.

Architecture of ANN could be described using following statements [24]. A set of operating interconected elements (perceptrons or nodes) represents an directed graph. Let us consider a set of input units x_i, i=1, 2, . . ., m, and a set of output units y_j, j=1, 2, . . ., n. Each node j performs a transfer function f_j of the form

$$y_j = f_j\Big(\sum_{i=0}^{m} w_{ji}x_i - \Theta_j\Big). \tag{5}$$

Connection weight between nodes i and j is denoted as w_{ji}, and threshold (or bias) of the node is denoted as Θ_j. Transfer function f_j is nonlinear, such as a heaviside, sigmoid, or Gaussian function.

For FNNs, sensitivity of hyperparameter exists in certain interval. Some adjusted parameters such as number of layers, number of perceptrons per layer, learning rate can be sensitive to the choice of presented set. Finding of the optimal set can be established after multiple simulations of learning procedure and prediction events.

Recurrent Neural Networks. RNNs are a part of ANNs designed for processing the sequential data. In a RNNs, at least one feedback connection must be present. The Hopfield model and the Boltzmann machine are two examples of RNNs. Dynamic and flexible structure of RNNs allows to handle sequences by maintaining a form of memory. Fig. 2 represents the schema of Recurrent Neural Network (RNN) where backpropagation in hidden layers was realized two times.

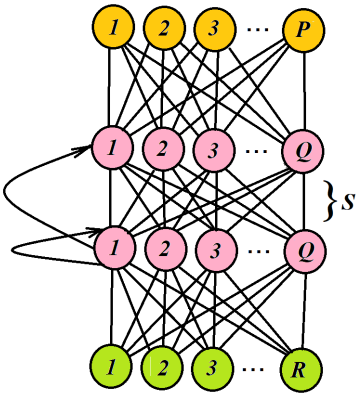


Fig. 2. Recurrent Neural Networks (RNNs).
Multiple hidden layers, S>1 (red): number of perceptrons Q,
input layer (yellow): number of units P,
output layer (green): number of units R.

Main behaviour of RNNs could be formulated as follows: hidden states for memory; special parameter sharing; flexibility for variable-length sequences [25-26].

The hidden state of the RNNs could be used as an internal memory for storage of information from previous steps. This hidden state is updated at each time step based on the current input and the previous hidden state. This realization allows the network to maintain context that evolves as the sequence progresses.

For RNNs, special parameter sharing is realized. RNNs use the same set of weights and biases across all time steps, enabling them to share parameters and learn to generalize patterns in sequences. This parameter sharing is a key aspect that allows RNNs to handle sequences of varying lengths.

Ability to process sequences of variable lengths represents the strengths of RNNs. RNNs can adapt to different lengths by dynamically updating their hidden states through time.

Limitations of RNNs could be formulated as the difficulty in capturing of long-term dependencies. However, advanced architectures like Long Short-Term Memory (LSTM) [27] and Gated Recurrent Unit (GRU) [28] can improve the modeling of longer-range dependencies in sequential data.

Modular Neural Networks. MNNs involve structuring the neural network architecture into modular components which serve a specific function. To create more complex network architecture these modules can be combined into traditional leyered structures. This design allows improving the interpretability, generalization, and efficiency of neural networks [29].

Main behaviour of RNNs could be formulated as follow: presence of domain-specific modules; high level of scalability and adaptability; interchangeable components; robustness and fault tolerance.

Depending on the specification of tasks, modular networks can be enriched by domain-specific modules (specialized modules) designed to handle particular types of data.

Due to modularity, high level of scalability and adaptable architecture is present. Networks can be scaled up by adding more modules, and modifications can be made by replacing or tweaking specific modules.

In modular networks, the individual modules are designed to be interchangeable. Due to replacing or relinking of different modules, various temporary configurations of entire network could be simulated when without redesigning the total network. This feature is very important for testing purposes [30].

The robustness and fault tolerance of neural networks could be essentialy enhanced due to presence of modularity. In case of failing of specific module, it can be adjusted without affecting the entire system. This makes modular networks more resilient to changes and uncertainties.

Variations of ANNs such as Artificial Nonmonotonic Neural Networks (ANNNs) are presented by Boutsinas et al [19]. ANNNs are a kind of hybrid learning systems that provide explanation to the trained neural networks for following goals: a) acquiring symbolic knowledge about a domain, b) improving that knowledge using a set of classifications examples, c) extracting comprehensible symbolic information.

2. Perceptron

Perceptrons are the building blocks of ANNs, and they serve as the fundamental unit for information processing. The essence of a perceptron lies in its simplicity and its ability to make binary decisions based on weighted inputs. A perceptron takes multiple binary inputs (0;1) and produces a single binary output. Each input is associated with a weight that represents the importance of that input. The perceptron computes a weighted sum of its inputs, and this sum is passed through an activation function. The most common activation function for a perceptron is a step function, where the output is 1 if the weighted sum is above a certain threshold and 0 otherwise. Learning rules for perceptrons adjust the weights based on errors in classification. The perceptron learning algorithm helps the perceptron learn the optimal weights to improve its performance on a given task.

Two types of perceptrons are significant for ANN: single-layer perceptron (SLP) and multi-layer perceptron (MLP). SLP consists of only one layer of perceptrons. For linearly separable patterns, an effective SLP learning procedure allows solving the simple not complicated tasks (predictions with high probability). MLPs consist of several hidden layers of perceptrons. Presence of multiple hidden layers allows MLPs to learn complex, non-linear patterns. Modern neural networks are constructed using MLPs. However, Gómez-Ramos et al [6] showed that the MLP has important delimitations in initialization weights.

Transfer function f_j is nonlinear, such as a heaviside, sigmoid, or Gaussian function. Heaviside step function (unit step function) usually denoted by H is a function, the value of which is zero for negative arguments and one for positive arguments:

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{6}$$

Characteristic S-shaped curve or sigmoid curve could be generated using logistic function $\sigma(x)$

$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x} = 1 - \sigma(-x) \tag{7}$$

or hyperbolic tangent (shifted and scaled version of the logistic function)

$$f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{8}$$

Activation functions introduce non-linearities to the model, allowing it to capture complex relationships in the data. Common activation functions include sigmoid, hyperbolic tangent, and rectified

linear unit (ReLU) [40]:

$$f(x) = \max(0, x) = \frac{x + |x|}{2} = \begin{cases} x, & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Complete classification of activation functions was described by Jain et al [13]. Nejad [12] presented expanded list of common actication functions such as exponential, hyperbolic, sine, softmax..

3. Learning procedure

The operation of ANNs is divided into two stages: learning (training) and generalization (prediction, recalling). The training of the network should be done using prepared examples such as a set of input parameters and a set of output parameters associated with multiple events. After learning procedure provided using selected learning algorithm, network parameters are adapted. Network is ready for predictions.

In ANNs, learning procedure represents modification and adjustments of the weights of links between the nodes of a graph. Learning in ANN can be classified into three categories: supervised learning, unsupervised learning, and reinforcement learning.

Feedforward neural networks (FNNs) are trained using supervised learning, where the model is provided with input-output pairs, and the weights are adjusted iteratively to minimize the difference between predicted and actual outputs. Wang [4] accented importance of training improvement because ANN will get better results with the progress of training. The backpropagation (BP) algorithm is commonly used for training. It calculates the gradient of the error with respect to the weights and adjusts the weights to minimize the error. Basheera et al [12] presented related learning rules with special emphasis on BP of ANNs. Perceptron learning algorithms were described by Jain et al [14].

Two big learning problems exist: large amounts of data and overfitting in training. Tasks of FNNs often require a large amount of labeled data to generalize well. Training deep FNNs on small datasets can lead to overfitting, where the model performs well on the training data but poorly on unseen data. Deep FNNs with many parameters can be overfitted, especially when the training data set is limited. Adjustment techniques such as dropout or weight decay are commonly used to partial solution of this problem [31].

Limitations of learning procedure for FNNs are the following: limited interpretability and computational intensity. Understanding the decision-making process of FNNs can be complex and trivial. For deep neural networks, including FNNs, realisation is presented in form of black box. It is difficult to understand why a particular prediction was made, restrict their application to areas where interpretability is critical. Training procedure of deep FNNs can be computationally intensive, requiring large resources both processing time and computing power. If resources are incomplete this can be a serious limitation for applications.

Training of RNNs involves a variation of the backpropagation algorithm titled as Backpropagation Through Time (BPTT) [32-33]. BPTT is used to update the network's parameters by considering the unfolding of the network over time, effectively treating the sequence as an unfolded computational graph. For RNNs, training procedure could be to slow down due to vanishing and exploding gradient problems. These problems arise when the gradients become too small, resulting in slow learning or ineffective updates (disappearing), or too large causing digital instability (explosion). Werbos [32] reviewed BPTT method for pattern recognition and fault diagnosis. For speech recognition and understanding, Ismail [33] proposed a new type of RRN, which contains each output unit connected to itself and is also fully connected other output devices and all hidden units. This modification of RRN was used in parallel with BPTT was assessed as promising.

The basic equations for backpropagation through time, and applications to areas like pattern recognition involving dynamic systems, systems identification, and control are discussed. Further extensions of this method, to deal with systems other than neural networks, systems involving simultaneous equations, or true recurrent networks, and other practical issues arising with the method are described. Pseudocode is provided to clarify the algorithms. The chain rule for ordered derivatives-the theorem which underlies backpropagation-is briefly discussed. The focus is on designing a simpler version of backpropagation which can be translated into computer code and applied directly by neutral network users

For MNNs, improved training and transfer learning could be organised in modular networks in comparison to others. Training of this type allows to reuse these modules in different contexts or combine them for transfer learning. This is especially useful when working on related tasks.

Farizawani et al [34] described the Conjugate Gradient (CG) technique as one of the most popular optimization practices used in ANN. Nowadays, CG technique allows improving the learning algorithm.

Training strategy sometimes could be titled asnon adequite. Walczak et al [11] describes the problem of ANN architecture when exact or approximate number of hidden units is required to model The approach consists of three steps.

1. Initially create a architecture of ANN with a very small or very large number of hidden units.
2. Train the network for some predetermined number of epochs.
3. Evaluate the error of the output nodes.

Singh et al [35] analyses the behaviour of training data. Longer time series of training samples will contain more events of different types, and hence, the generalization ability of the ANN will improve. However, for long time series, if considerable repetition of the same type of information is present, the ANN may not become "wiser", and one may be just wasting computational effort and time.

4. Predictions using ANNs

Basheera et al [12] represents history of the evolution of neurocomputing and its relation to the field of neurobiology.

Finance. Nejad [13] presented applications of ANNS by following categories: prediction, classification, data associacion, data conceptualization, data filtering. Jaiswal et al [15] used backpropagation neural network for predicting future prices by modifying these techniques as per requirements.

Classification. Nejad [12] emphasized pattern recognition as the main application areas of biological and artificial neural networks. Petridis et al [36] presented an recursive classifier of Incremental Credit Assignment (ICRA) type appropriate for online time series classifications.

Natural language processing. Feedforward neural networks (FNNs) are versatile and applied in various domains, including image and speech recognition, natural language processing [5], and regression tasks. For convolutional neural networks (CNNs) and recurrent neural networks (RNNs), FNNs are the building blocks for more complex architectures. RNNs find applications in tasks involving natural language processing (NLP), time series analysis, speech recognition, and other sequential data problems [4].

Education. Usage of ANN for educational purposes [37] shows several advantages over human teachers. ANN based education is more attractive for online learning. AI system dedicated personally allows adapting the teaching speed and satisfaction of individual needs (in context of subject). Scalability looks as the first prerogative. Availability at any time and any place looks as the second prerogative.

When learning a foreign language, language practice is very important. Speaking a foreign language with a robot can be more convenient for the beginning speaker. Implementing AI in education systems reduces the student's shame of the initial speach failure.

Autonomious education agents work best on constrained topics like math and programming where good or satisfied answers can be easily identified. Such agents will not be able to appreciate beauty poem or rate the novelty.

Industry. Bakas et al [38] presented the study which explores the use of ANNs on a heat transfer application. Yildirim et al [17] presented the study which compares non-linear regression and ANN models in predicting 10 yarn properties shaped by the influence of winding speed, quenching air temperature and/or quenching air speed during production. A multilayer perceptron ANN model was created by training 81 patterns using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm described in Ref. [39]. The hyperbolic tangent, or $\tanh$, activation function and logistic activation functions were used for the hidden and output layers respectively. ANN simultaneously predicted all of the 10 final properties of a yarn: tensile strength, tensile strain, draw force, crystallinity ratio, dye uptake based on the colour strengths, brightness, boiling shrinkage and yarn evenness, ANN predictions are more accurately than the non-linear regression model. Ali [39] described the BFGS method for large-scale optimization problems. More et al [15] presented ANN techniques to optimise and forecast mine water treatment plant parameters.

Biology, medicine. Basheera et al [12] described several practical application, where ANNs were used to model the microbial growth curves. The developed model was reasonably accurate in simulating both training and test time-dependent growth curves as affected by temperature and pH. Yacine et al [18] described a novel ANN approach based on an Adaptive Riemannian Kernel (ARK-ANN), to classify Electroencephalographic (EEG) motor imaging signals in the context of Brain Computer Interface (BCI). A multilayer perceptron is used to classify the covariance matrices of Motor Imagery (MI) signals employing an adaptive optimization of the testing set. Increasing of precision by 8.2% in comparison to the SVM based method was fixed.

OCR, vision. Possible solutions of complicated tasks related to the Optical character recognition (OCR) were described by Jain et al [14]. In 1969, Fukushima [40] described the pioneering realization of visual feature extracting network for handwritten character recognition. The design of the network was suggested by visual systems of cat. The network is composed of analog threshold elements connected in layers. Each analog threshold element receives inputs from a large number of elements in the neighboring layers and performs its own special functions. This successfully implemented method of information processing based on biological analogues inspired further constructions of complex systems.

Conclusions

1. Artificial Neural Networks (ANN) currently implementing Feedforward Neural Network (FNN) and Recurrent Neural Network (RNN) are very useful for applied tasks related to OCR, speech recognition, classification, stock market prediction, etc. This versatility is achieved due to FNN's ability to approximate any continuous function under certain conditions.
2. ANN learning procedure was demonstrated using several cuts: problems related to ANN architecture and perceptron dynamics, interpretation problems of overfitting and possible methods to improve this situation.
3. The advantages of prediction using ANNs have been demonstrated in several popular fields such as adaptive educology, classification in medicine and biology, industry, etc.

Abbreviations

AI	-	Artificial Intelligence
ARK	-	Adaptive Riemannian Kernel
ANN	-	Artificial Neural Network
ANNN	-	Artificial Nonmonotonic Neural Network
BCI	-	Brain Computer Interface
BP	-	Backpropagation
BPTT	-	Backpropagation Through Time
BFGS	-	Broyden-Fletcher-Goldfarb-Shanno
CG	-	Conjugate Gradient
CNN	-	Convolutional Neural Network
EEG	-	Electroencephalographic
FNN	-	Feedforward Neural Network
GRU	-	Gated Recurrent Unit
ICRA	-	Incremental Credit Assignment
LSTM	-	Long Short-Term Memory
ML	-	Machine Learning
MLP	-	Multi-Layer Perceptron
MNN	-	Modular Neural Network
NLP	-	Natural Language Processing
OCR	-	Optical character recognition
RNN	-	Recurrent Neural Network
SVM	-	Support Vector Machine

References

1. Soori, M.; Arezoo, B.; Dastres, R. (2023) Artificial Intelligence, Machine Learning and Deep Learning in Advanced Robotics, A Review. - *Cognitive Robotics* 3(2023)54-70. - https://doi.org/10.1016/j.cogr.2023.04.001.
2. Bartneck, C.; Lütge, C.; Wagner, A.; Welsh, S. (2021) Capter 2. What Is AI? In: An Introduction to Ethics in Robotics and AI - Springer - https://doi.org/10.1007/978-3-030-51110-4
3. Badue, C.; Guidolini, R.; Carneiro, R.; Azevedo, P.; Cardoso, V.; Forechi, A.; Jesus, L.; Berriel, R.; Paixão, T.; Mutz, T.; Veronese, L.; Oliveira-Santos, T.; De Souza, A. (2021) Self-driving cars: A survey - *Expert Systems with Applications* - 165(2021)113816 - https://doi.org/10.1016/j.eswa.2020.113816.
4. Wang, X. (2018) Overview of the Research Status on Artificial Neural Networks - *Advances in Social Science, Education and Humanities Research (ASSEHR)* 130(2018)351-356.
5. Sodhar, I.; Buller, A. (2020). Natural Language Processing: Applications, Techniques and Challenges. - In: Adavances In Computer Science - AkiNik Publications New Delhi, 2020. - https://doi.org/10.22271/ed.book.784-.
6. Gómez-Ramos, E.; Venegas-Martínez, F. (2013) A Review of Artificial Neural Networks: How Well Do They Perform in Forecasting Time Series? - *Analitika, Revista de analisis estadistico* 3 (2013)7-15.

7. Ahmed, R. (2017) Applications of Artificial Neural Networks in E-Learning Personalization - *International Journal of Computer Applications* 158 (2017) 2.
8. Zhou, Y.; Huang, C.; Hu, Q.; Zhu, J.; Tang, Y. (2018) Personalized learning full-path recommendation model based on LSTM neural networks - *Information Sciences* 444(2018) 135-152 - https://doi.org/10.1016/j.ins.2018.02.053.
9. Zapalska, A.; Brozik, D. (2006) Learning styles and online education - *Campus-Wide Information Systems* 23(2006) 325-335 - https://doi.org/10.1108/10650740610714080.
10. Larsen, J. (1999) Introduction to Artificial Neural Networks. - Lingby: Technical university of Denmark.
11. Walczak, S.; Cerpa, N. (2003). Artificial Neural Networks. -631-645 In: Encyclopedia of Physical Science and Technology - https://doi.org/10.1016/B0-12-227410-5/00837-1.
12. Basheera, I.A.; Hajmeerb, M. (2000) Artificial neural networks: fundamentals, computing, design, and application - *Journal of Microbiological Methods* 43(2000)3–31.
13. Nejad, A. (2020) Evolutionary Models for Adaptive Artificial Neural Networks in Accounting and Finance Trends. - Academia Eductional, 2020. - Series: Evolutionary Models for Adaptive ANNs - https://doi.org/10.198769124.
14. Jain, A.; Mao, J.; Mohiuddin, K. (1996) Artificial neural networks: a tutorial - *Computer* 29(1996)31-44 - https://doi.org/10.1109/2.485891.
15. Jaiswal, J.; Das, R. (2018) Artificial Neural Network Algorithms based Nonlinear Data Analysis for Forecasting in the Finance Sector - *International Journal of Engineering and Technology(UAE)* 7(2018)169-176 - https://doi.org/10.14419/ijet.v7i4.10.20829.
16. More, K,; Wolkersdorfer, C. (2023) Application of machine learning algorithms for nonlinear system forecasting through analytics - A case study with mining influenced water data - *Water Resources and Industry* 29(2023)100209 - https://doi.org/10.1016/j.wri.2023.100209.
17. Yıldırım, K.;, Ogut, H.; Ulcay, Y. (2017) Comparing the Prediction Capabilities of Artificial Neural Network (ANN) and Nonlinear Regression Models in PetPoy Yarn Characteristics and Optimization of Yarn Production Conditions - *Journal of Engineered Fibers and Fabrics* 12(2017)7.
18. Yacine, F.; Salah, H.; Amar, K.; Ahmad, K. (2023) A novel ANN adaptive Riemannian-based kernel classification for motor imagery - *Biomedical Physics & Engineering Express* 9(2023) 015010 - https://dx.doi.org/10.1088/2057-1976/acaca2.
19. Boutsinas, B.; Vrahatis, M.N. (2001) Artificial nonmonotonic neural networks - *Artificial Intelligence* 132(2001)1-38 - https://doi.org/10.1016/S0004-3702(01)00126-6.
20. Kohli, S.; Miglani, S.; Rapariya, R. (2014) Basics of Artificial Neural Network - *International Journal of Computer Science and Mobile Computing* 3(2014) 745-751.
21. Bekesiene, S.; Smaliukiene, R.; Vaicaitiene, R. (2021) Using Artificial Neural Networks in Predicting the Level of Stress among Military Conscripts - *Mathematics* 9(2021) 626 - https://doi.org/10.3390/math9060626.
22. Zakaria, M.; AL-Shebany, M.; Sarhan, S. (2014) Artificial Neural Network : A Brief Overview - *International Journal of Engineering Research and Applications* 4(2014)07-12
23. Kratsios, A.; Bilokopytov, E. (2020) Non-Euclidean Universal Approximation - *Advances in Neural Information Processing Systems* 33(2020).
24. Yao X. (1999) Evolving Artificial Neural Networks - *Proceedings of the IEEE* 87 (1999)1423
25. Du, K.; Swamy, M.N.S. (2014) Recurrent Neural Networks. - In: Neural Networks and Statistical Learning Springer, London, 2014. - https://doi.org/10.1007/978-1-4471-5571-3_11.
26. Rishickesh, R.; Ram Kumar, R.P.; Shahina, A.; Nayeemullah Khan, A. (2019) Identification of Duplication in Questions Posedon Knowledge Sharing Platform Quora using Machine Learning Techniques - *International Journal of Innovative Technology and Exploring Engineering (IJITEE)* 8(2019)2444 - https://doi.org/10.35940/ijitee.
27. Jurafsky, D.; Martin, J. (2023) Chapter 9. RNNs and LSTMs. - In: Speech and Language Processing. - https://web.stanford.edu/ jurafsky/slp3/.
28. Abumohsen, M.; Owda, A.; Owda, M. (2023) Electrical Load Forecasting Using LSTM, GRU, and RNN Algorithms - *Energies* 16(2023) 2283 - https://doi.org/10.3390/en16052283.
29. Auda, G.; Kamel, D. (1999) Modular neural networks: A survey - *International Journal of Neural Systems* 9(1999)129-151.
30. Shukla, A.; Tiwari, R.; Kala, R. (2010) Modular Neural Networks. - In: Towards Hybrid and Adaptive Computing. Studies in Computational Intelligence, vol 307. - Springer, Berlin, Heidelberg. - https://doi.org/10.1007/978-3-642-14344-1_14
31. Zhu, Z.; Liu, F.; Chrysos, G.; Locatello, F.; Cevher, V. (2023) Benign Overfitting in Deep Neural Networks under Lazy Training - *Proceedings of the 40th International Conference on Machine Learning*, Honolulu, Hawaii, USA - PMLR 202, 2023.
32. Werbos, P. (1990) Backpropagation through time: what it does and how to do it - *Proceedings of the IEEE* 78(1990)1550-1560 - https://doi.org/10.1109/5.58337.
33. Ismail, S.; Ahmad, A. (2004) Recurrent neural network with backpropagation through time algorithm for Arabic recognition - *Proceedings 18th European Simulation Multiconference*, SCS Europe.
34. Farizawani, A.G.; Puteh, M.; Marina, Y.; Rivaie, A. (2020) A review of artificial neural network learning rule based on multiple variant of conjugate gradient approaches - *Journal of Physics: Conference Series* 1529(2020)022040 - https://doi.org/10.1088/1742-6596/1529/2/022040
35. Singh, S.; Sharad, J.; András, B. (2010) Training of Artificial Neural Networks using Information-Rich Data - *Hydrology* 1(2010) 40-62. - https://doi.org/10.3390/hydrology1010040.
36. Petridis, V.; Kehagias, A. (1996) A Recurrent Network Implementation of Time Series Classi
cation - *Neural Computation* 8(1996)357-372.
37. Bartneck, C.; Lütge, C.; Wagner, A.; Welsh, S. (2021) Capter 9 . Application areas of AI. - In: An Introduction to Ethics in Robotics and AI - Springer - https://doi.org/10.1007/978-3-030-51110-4
38. Bakas, I.; Kontoleon, K.J. (2021) Performance Evaluation of Artificial Neural Networks (ANN) Predicting Heat Transfer through Masonry Walls Exposed to Fire - *Applied Science* 11 (2021)11435 - https://doi.org/10.3390/app112311435.
39. Ali, M. (2021) Modified limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm for unconstrained optimization problem - *Indonesian Journal of Electrical Engineering and Computer Science* 24(2021)1027 - https://doi.org/10.11591/ijeecs.v24.i2.pp1027-1035.
40. Fukushima, K. (1969) Visual Feature Extraction by a Multilayered Network of Analog Threshold Elements - *IEEE Transactions on Systems Science and Cybernetics* 5(1969)322-333 - https://doi.org/10.1109/TSSC.1969.300225.

Article

Studying the Recurrent Sequence Generated by Power Function using QUATTRO-20

Jelena Kozmina*, Alytis Gruodis
Department of IT, Vilnius Business College, Kalvarijų g. 129-401, LT-08221 Vilnius, Lithuania
* Corresponding author, e-mail: *jelena.kozmina@gmail.com*

Received: 4 November 2023
Accepted: 10 December 2023
Online: 30 December 2023

Abstract. We presented the bifurcational diagram of power function $\Phi(x) = r \cdot x \cdot (1 - x^2)$ which could be treated as first approximation of trigonometric function $F(x) = r \cdot x \cdot \cos x$. Using second composite $\Phi^2(x)$ in analytical form and solving 8-th degree polynomial equation bifurcational diagram with period doubling 1, 2, 4 was obtained and attractors were established. Analytical solutions of expressions $x = \Phi^2(x)$ allows us to establish the fixed point attractors and periodic attractors in interval (-$\sqrt{5}$, $\sqrt{5}$). Bifurcation diagram obtained analytically was compared with its aproximate analogue Finite State diagram.

JEL: C25.

Keywords: recurrent sequence; power function; fixed point attractor; periodic attractor; Finite State Diagram; Bifurcation diagram; index of Lyapunov exponent; CobWeb plot.

Citation: Jelena Kozmina, Alytis Gruodis (2023) Studying the Recurrent Sequence Generated by Power Function using QUATTRO-20. – *Applied Business: Issues & Solutions* 2(2023)28–36 – ISSN 2783-6967. https://doi.org/10.57005/ab.2023.2.4

Introduction

Recurrent sequences can be important in various fields of physics, chemistry, engineering, and informatics in the context of modeling of periodic or oscillatory behavior which orccur at visualization of reaction or engine cycle, cipher/key generation, big data maintening etc.

Trigonometric functions (sine and cosine) naturally represent an oscillatory behavior (vibrations, waves, etc) with periodic components. Phenomenon of recurrence allows describing the evolution of complex systems in several dynamical regimes where the characteristic regime parameters repeat or oscillate over time [1, 2]. This is particularly relevant in audio processing, where signals often exhibit the repeating patterns.

Recurrent sequences with trigonometric functions can arise in solutions to certain differential equations. These equations may model physical phenomena where both recurrence and periodicity play a significant role [3, 4].

Typical tasks of BigData requires the analysis of data collections, where data sampling was provided over time. Usage of recurrent functions with trigonometric terms can help capture patterns and trends that repeat at regular/pseudoregular intervals [5, 6].

Usage of recurrent sequencies with trigonometric terms in Artificial Neural Network and Machine Learning could be estimated as the novel implementation of processing of the data sequences [6-9]. Trigonometric functions can be incorporated into the activation functions or hidden states of Artificial Neural Networks to represent the time-domain or temporal dependencies with periodic characteristics.

Analysis technique is known [10-12]: Finite State Diagram, Bifurcation diagram, distribution of index of Lyapunov exponent, CobWeb plot for different composites of function $F(x)$: $F^2(x)$, $F^3(x)$, $F^4(x)$. This solution pathway was shown for classical Verhulst equation $F(x) = r \cdot x \cdot (1 - x)$ presented in Refs. [10, 11] as standard manipulation in analytical form using polinomial behaviour. Unfortunately, for recurrent trigonometric functions, possibility to express the composites of high order F^2, F^3, F^4 could be realized in approximate form only.

This work is devoted to understanding the specific context in which recurrent sequences generated by the trigonometric functions $F(x) = r \cdot x \cdot \cos x$ could be replaced with corresponding polynomial function $\Phi(x) = r \cdot x \cdot (1 - x^2)$ Depending on the field and application, these functions can provide accurate and efficient dynamical representations of real-world phenomena with periodic behavior.

1. Literature review

Recurrent relations. Recurrent relation expresses the relationship between the terms of a sequence, when the previous term predeterms the behaviour of the current. Usage of the mentioned recursive technique is popular due to simplicity to control the generation. Dosly et al [13] presented so called trigonometric transformation technique for recurrence relations. Brooke et al [14] described several forms of second-order linear Recurrent relation, which satifies the requested features: jump from a sequence with period k to second-order linear Recurrent relation. In that case, three distinct non-trivial periods will be generated. Farris et al [15] proved, that for certain families of functions f and g, a sequence generated by a recurrent relation

$$a_{n+1} = f(n) \cdot a_n + g(n) \cdot a_{n-1} \tag{10}$$

is Benford for all initial values. Importance of Benford law [16] is well known. It describes a natural phenomenon from many real-world datasets. It reflects a pattern that emerges in naturally occurring numerical data, such as physical constants, parameters of biological systems, etc. By analysing this distribution, it is possible to distinguish between sequences related to the naturally occuring system and formula-generated data.

Indicators of chaos. Generally, pure chaotic manner of recurrent relation could be established from natural processes only [16]. As mathematical models, the Lorenz system, the Henon map as well as logistic map are well known [17]. Each of these models has its own set of equations. Andrianov et al [17] use various variants of Verhulst-like ordinary differential equations and ordinary difference equations. Several examples of deterministic discretization and chaotic continualization (procedure is based on Padé approximants) are analysed. Characteristic parameters of chaotic dynamical systems as the Lyapunov exponents and the Lyapunov dimensions were presented and discussed.

Gutierrez et al [18] analyzed Verhulst logistic equation and a couple of forms of the corresponding logistic maps. In particular, they presented the map

$$x_{n+1} - x_n = r \cdot x_n \cdot (1 - x_{n+1}) \tag{11}$$

or in usual form:

$$x_{n+1} = \frac{(1+r)x_n}{1+r\cdot x_n} = 1 - \frac{1-x_n}{1+r\cdot x_n} \tag{12}$$

which is identical to the logistic equation from the standpoint of the general Riccati solution.

Ragulskis et al [19] presented the concept of the Hankel rank of a solution of the discrete nonlinear dynamical system. Computation of ranks of subsequences of solutions helps to identify and assess the sensitivity of the system to initial conditions: stability or convergence properties.

Hikihara et al [20] presented explicit historical review of the systems of deterministic chaos. They claim that unpredictability is due to sensitive dependence on initial conditions caused by rapid divergence of neighbouring solutions. This property is quite common in nonlinear differential equations with three or more variables, invertible maps in two or more dimensions, and all non-invertible maps. Chaotic dynamics is locally expansive in one or more directions in phase space and contractive in the remaining dimensions. Only in the last 50 years, the significant applications are presented for investigation in the biological and life sciences. Analytical, geometrical, and computational methods have been developed to detect and characterize chaotic sets, and experiments have confirmed that they appear in a variety of real systems.

Ditto et al [21] presented description if investigations in chaotic system, in particular, problems of Artificial Intelligence, such as Artificial Neural Network, fuzzy logic, and genetic algorithms can be employed together with chaotic systems (e.g., neural networks and chaos, or neural networks and fuzzy logic and chaos.) Components of hybrid systems complement each other creating new approaches to solve problems.

Nosrati et al [22] investigated the biological systems in the real world using the singular system theory and fractional calculus. Some fractional-order singular biological systems are established, and some qualitative analyses of proposed models are performed. It was established that presence of fractional order changes the stability of the solutions and enriches the dynamics of system. In comparison to standard model, fractional-order singular biological systems exhibit instability phenomena (in term of bifurcation).

Generally, classic chaos-detection tools are highly sensitive to measurement noise. Toker et al [23] presented novel tool which combine several classical tools into an automated processing pipeline, and show that this pipeline can detect the presence (or absence) of chaos in noisy recordings, even for difficult cases.

Fehrle et al [24] presented polynomial chaos expansion method for series expansion of uncertain model inputs. Suitability of polynomial chaos expansion for computational economics is discussed.

Cryptography. Recurrent relations are important in cryptographic applications because the generation of pseudorandom numbers play a crucial role in cryptographic protocols and algorithms. Estimation of sequence quality of pseudorandom numbers [25] must be done due to security restrictions. Alawida et al [26] presented a hybrid chaotic which uses cascade and combination methods as a nonlinear chaotification function. Analysis shows that enhanced maps have a larger chaotic range, low correlation, uniform data distribution and better chaotic properties. Several simple pseudorandom number generators are designed based on a classical map and its enhanced variant.

2. Sequence generated by power function

We would like to study the properties of sequence

$$x_{t+1} = F(x_t), \quad t = 0, 1, 2, \dots , \tag{13}$$

where

$$F(x) = r\cdot x\cdot \cos x \tag{14}$$

for different values of parameter r. Anticipating the difficulties awaiting us, let us start with the approximation of the function $\cos x$ using MacLaurin series. The explicit form the Maclaurin series of a function $\cos x$ is presented below:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots . \tag{15}$$

The Maclaurin series is used to create a polynomial function

$$\cos x \approx 1 - \frac{x^2}{2}. \tag{16}$$

Using polynomial function $G(x)$

$$G(x) = r\cdot x\cdot \left(1 - \frac{x^2}{2}\right) \tag{17}$$

we generate the sequence

$$x_{t+1} = r\cdot x_t\cdot \left(1 - \frac{x_t^2}{2}\right), \quad t = 0, 1, 2, \dots . \tag{18}$$

Transforming Eq.(18)

$$\frac{x_{t+1}}{\sqrt{2}} = r\cdot \frac{x_t}{\sqrt{2}}\cdot \left(1 - \left(\frac{x_t}{\sqrt{2}}\right)^2\right) \tag{19}$$

and using the substitution

$$\frac{x_t}{\sqrt{2}} \to x_t \tag{20}$$

leads us to the sequence

$$x_{t+1} = r\cdot x_t\cdot (1 - x_t^2), \tag{21}$$

or

$$x_{t+1} = \Phi(x_t), \tag{22}$$

where

$$\Phi(x) = r\cdot x\cdot (1 - x^2). \tag{23}$$

The function $\Phi(x)$ reaches the maximum value $\frac{2r}{3\sqrt{3}}$ on the segment [–1,1] at the point x=$\frac{1}{\sqrt{3}}$ and the minimum value $-\frac{2r}{3\sqrt{3}}$ on the segment [–1,1] at the point x=$-\frac{1}{\sqrt{3}}$. So function $\Phi(x)$:[–1,1]→[–1,1] if

$$\left|\frac{2r}{3\sqrt{3}}\right| \le 1. \tag{24}$$

This fact imposes restriction on r:

$$r \in \left[-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right], \qquad \frac{3\sqrt{3}}{2} \approx 2.598 < 2.6. \tag{25}$$

3. Fixed point attractors

We know [10,12] that if we want to find fixed point attractor of the sequence x_{t+1}=$\Phi(x_t)$ then we have to solve the equation x=$\Phi(x)$. The function $\Phi(x)$ has to be a contracting map in a closed interval and absolute value of the derivative of function $\Phi(x)$ calculated at the solution of equation $x = \Phi(x)$ have to be less then 1.
We find the fixed points of function $\Phi(x)$ by solving the equation

$$x = r\cdot x\cdot (1 - x^2). \tag{26}$$

This equation has three solutions:

$$x = 0, \quad x = \pm\sqrt{\frac{r-1}{r}}. \tag{27}$$

Now we calculate the derivative of $\Phi(x)$:

$$\frac{d\Phi(x)}{dx} = r \cdot (1 - 3x^2). \tag{28}$$

At point x=0

$$\left|\frac{d\Phi}{dx}\Big|_{x=0}\right| = |r|. \tag{29}$$

If $|r|<1$ then point x=0 must be titled as fixed point attractor. Another two solutions of equation $x=\Phi(x)$ are $x=\pm\sqrt{\frac{r-1}{r}}$.

$$\left|\frac{d\Phi}{dx}\Big|_{x=\pm\sqrt{\frac{r-1}{r}}}\right| = \left|r \cdot \left(1 - 3\frac{r-1}{r}\right)\right| = \left|3 - 2r\right|, \tag{30}$$

$$|3 - 2r| < 1 \quad \Leftrightarrow \quad 1 < r < 2. \tag{31}$$

It means that points

$$x = \pm\sqrt{\frac{r-1}{r}} \tag{32}$$

are the fixed point attractors for $1<r<2$. Note if r=1.5 then

$$x = \pm\sqrt{\frac{r-1}{r}} = \pm\sqrt{\frac{1.5-1}{1.5}} = \pm\sqrt{\frac{1}{3}} \approx \pm 0.57735269 \tag{33}$$

and

$$\left|\frac{d\Phi}{dx}\Big|_{x=\pm\sqrt{\frac{1}{3}}}\right| = 3 - 2 \cdot 1.5 = 0. \tag{34}$$

Figs. 1, 2, 3 represent graphical solutions of two functions: $y = x$ and $y = \Phi(x)$ at different r: r=1.5, r=1.9, r=2.2 respectively. For r=1.5 (Fig. 1), three crossing points are present, but only two of them $x=\pm\sqrt{\frac{1}{3}}\approx 0.58$ are fixed point attractors (tangent lines are parallel to x axis). For r=1.9 (Fig. 2), three crossing points are present, but only two of them $x=\pm\sqrt{\frac{9}{19}}\approx 0.69$ are fixed point attractors. (tangent lines are not parallel to x axis). For r=2.2 (Fig. 3), three crossing points are present, including $x=\pm\sqrt{\frac{6}{11}}\approx 0.74$, but none of them are fixed point attractors. Value r=2.2 is out of previously established interval $1<r<2$.

4. Periodic attractors

We know that

$$x_{t+2} = \Phi(x_{t+1}) = \Phi(\Phi(x_t)) = \Phi^2(x_t), \tag{35}$$

where

$$\Phi(x) = r \cdot x \cdot (1 - x^2) = r \cdot (x - x^3). \tag{36}$$

Let us express the second order composite $\Phi^2(x)$:

$$\Phi^2(x) = \Phi(\Phi(x)) = r \cdot \left(r \cdot (x - x^3) - \left(r \cdot (x - x^3)\right)^3\right) = \tag{37}$$

$$= r^2 \cdot x \cdot (1 - x^2) \cdot \left(1 - r^2 \cdot x^2 \cdot (1 - x^2)^2\right). \tag{38}$$

Let us find the fixed points of the function $\Phi^2(x)$. To find them we solve equation $x=\Phi^2(x)$:

$$x = r^2 \cdot x \cdot (1 - x^2) \cdot \left(1 - r^2 \cdot x^2 \cdot (1 - x^2)^2\right). \tag{39}$$

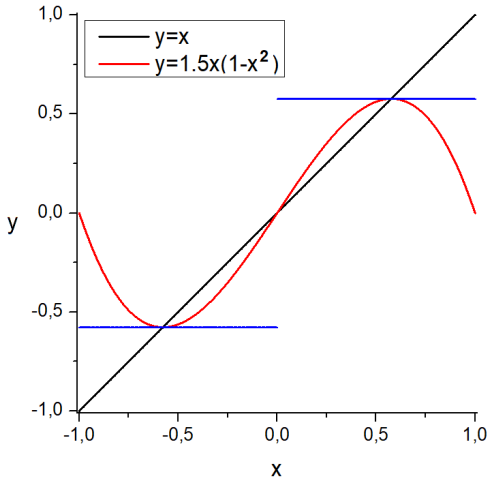


Fig. 1. $y = x$ and $y = \Phi(x)$ at r=1.5 (interval $1<r<2$). Three crossing points, but only two of them (approximate values – 0.58 and 0.58) are fixed point attractors (tangent lines are parallel to x axis).

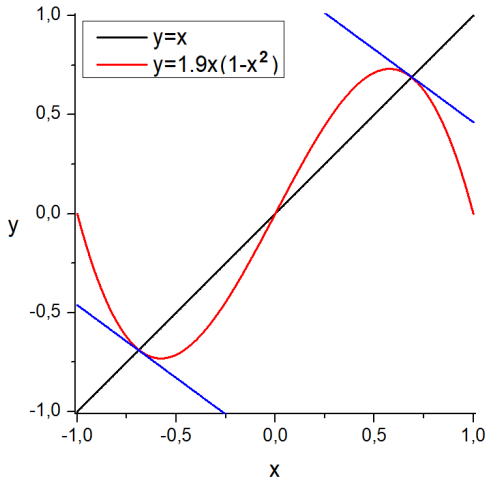


Fig. 2. $y = x$ and $y=\Phi(x)$ at r=1.9 (interval $1<r<2$). Three crossing points, but only two of them (approximate values – 0.69 and 0.69) are fixed point attractors.

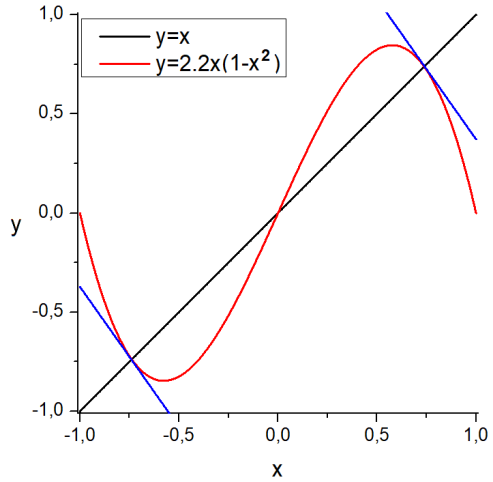


Fig. 3. $y = x$ and $y = \Phi(x)$ at r=2.2 (out of interval $1<r<2$). Three crossing points, but none of them are fixed point attractors.

Table 1. Horner's scheme for z=r–1.

	1	–3r	3r^2	–r^3–r	r^2–1
r–1	↓	+ r–1 =	+ –2r^2+r+1 =	+ r^3–1 =	+ –(r^2–1) =
	1	–2r–1	r^2+r+1	–r–1	0

Table 2. Horner's scheme for z=r+1.

	1	–2r–1	r^2+r+1	–r–1
r+1	↓	+ r+1 =	+ –r^2–r =	+ r+1 =
	1	–r	1	0

One solution is x=0. To get the other solutions we have to solve the equation

$$r^2 \cdot (1-x^2) \cdot \left(1 - r^2 \cdot x^2 \cdot (1-x^2)^2\right) = 1. \tag{40}$$

We transform it to the form of 8^{th}-power equation:

$$r^2 \cdot (1-x^2) \cdot \left(1 - r^2 \cdot x^2 \cdot (1-2x^2+x^4)\right) = 1, \tag{41}$$

$$(1-x^2) \cdot \left(r^2 - r^4x^2 + 2r^4x^4 - r^4x^6\right) = 1, \tag{42}$$

$$r^4x^8 - 3r^4x^6 + 3r^4x^4 - r^2(r^2+1)x^2 + r^2 - 1 = 0. \tag{43}$$

We know if $\Phi(x) = x$ then

$$\Phi^2(x) = \Phi(\Phi(x)) = \Phi(x) = x. \tag{44}$$

It means that the solutions of the equations $\Phi(x) = x$ are the solutions of the equations $\Phi^2(x)$=x. Let us use substitution rx^2=z, then we have to solve equation

$$z^4 - 3rz^3 + 3r^2z^2 - r(r^2+1)z + r^2 - 1 = 0. \tag{45}$$

$\Phi(x) = x$ for $x = \pm\sqrt{\frac{r-1}{r}}$, hence $z = rx^2 = r-1$ is the root of the equation

$$z^4 - 3rz^3 + 3r^2z^2 - r(r^2+1)z + r^2 - 1 = 0. \tag{46}$$

We use Horner's scheme presented in Table 1 and get the third power equation

$$z^3 - (2r+1)z^2 + (r^2+r+1)z - r - 1 = 0. \tag{47}$$

Let us check $z = r+1$ using Horner's scheme again - see Table 2. Having applied the Horner scheme we were convinced that $z = r+1$ is a solutions of Eq.(47). Now we have to solve the second power equation

$$z^2 - rz + 1 = 0. \tag{48}$$

The solutions of this equations are

$$z = \frac{r \pm \sqrt{r^2-4}}{2}. \tag{49}$$

We got four solutions of equation

$$z^4 - 3rz^3 + 3r^2z^2 - r(r^2+1)z + r^2 - 1 = 0, \tag{50}$$

$$z = r-1, \quad z = r+1, \quad z = \frac{r \pm \sqrt{r^2-4}}{2}. \tag{51}$$

These four solutions give us eight solutions of Eq.(43). Note that we used substitution rx^2=z, hence x^2=z/r, and

$$x_{1,2} = \pm\sqrt{\frac{r-1}{r}}, \quad x_{3,4} = \pm\sqrt{\frac{r+1}{r}}, \quad x_{5,6,7,8} = \pm\sqrt{\frac{r \pm \sqrt{r^2-4}}{2r}}. \tag{52}$$

To check which of these points are attracttors we need to count the value of first derivative of $\Phi^2(x)$ and make sure that at each of these points belongs to the interval (-1,1). For function

$$\Phi^2(x) = r^2 \cdot \left((x - x^3) - r^2(x-x^3)^3\right) \tag{53}$$

let's calculate the first derivative with respect to x:

$$\frac{d\Phi^2(x)}{dx} = r^2 \cdot (1-3x^2) \cdot \Big(1 - 3r^2x^2 \cdot (1-x^2)^2\Big). \tag{54}$$

For solution $x_{1,2}$

$$x_{1,2}^2 = \frac{r-1}{r}, \qquad 1-3x_{1,2}^2 = \frac{3-2r}{r}, \qquad (1-x_{1,2}^2)^2 = \frac{1}{r^2}, \qquad 1-3r^2x^2 \cdot (1-x^2)^2 = \frac{3-2r}{r}, \tag{55}$$

then

$$\frac{d\Phi^2(x)}{dx}\Big|_{x=\pm\sqrt{\frac{r-1}{r}}} = (3-2r)^2. \tag{56}$$

The first derivative of $\Phi^2(x)$ calculated at points $x_{1,2}$ have to be less than 1:

$$\frac{d\Phi^2(x)}{dx}\Big|_{x=\pm\sqrt{\frac{r-1}{r}}} = (3-2r)^2 < 1. \tag{57}$$

This takes place if $r\in(1,2)$. Thus we have established that points

$$x_{1,2} = \pm\sqrt{\frac{r-1}{r}} \tag{58}$$

are fixed point attractors if $r\in(1,2)$.
Solutions

$$x_{3,4} = \pm\sqrt{\frac{r+1}{r}} \tag{59}$$

is defined if $r<-1$. Similarly we can calculate that

$$\frac{d\Phi^2(x)}{dx}\Big|_{x=\pm\sqrt{\frac{r+1}{r}}} = (3+2r)^2 < 1 \tag{60}$$

if $r\in(-2,-1)$. Hence, points $x_{3,4}$ are fixed point attractors if $r\in(-2,-1)$.
To simplify the assessment of the derivative of the function $\Phi^2(x)$ at the points

$$x_{5,6,7,8} = \pm\sqrt{\frac{r \pm \sqrt{r^2-4}}{2r}} \tag{61}$$

we will use Eq.(40)

$$r^2 \cdot (1-x^2) \cdot (1 - r^2 \cdot x^2 \cdot (1-x^2)^2) = 1 \tag{62}$$

from which these solutions were obtained. We can transform this equation

$$-r^2 \cdot x^2 \cdot (1-x^2)^2 = \frac{1}{r^2 \cdot (1-x^2)} - 1 \tag{63}$$

and substitute it into expression of first derivative of $\Phi^2(x)$ with respect to x. For $x = x_{5,6,7,8}$ we get

$$\frac{d\Phi^2(x)}{dx}\Big|_{x=\pm\sqrt{\frac{r\pm\sqrt{r^2-4}}{2r}}} = r^2 \cdot (1-3x^2) \cdot (1 - 3r^2x^2 \cdot (1-x^2)^2) = \tag{64}$$

$$= r^2 \cdot (1-3x^2) \cdot \Big(\frac{3}{r^2 \cdot (1-x^2)} - 2\Big) = \tag{65}$$

$$= \frac{(1-3x^2)(3-2r^2+2r^2x^2)}{1-x^2}. \tag{66}$$

We have to solve the inequality

$$\left|\frac{d\Phi^2(x)}{dx}\Big|_{x=\pm\sqrt{\frac{r\pm\sqrt{r^2-4}}{2r}}}\right| < 1. \tag{67}$$

If $x\in(-1,1)$ then $1-x^2>0$. For $x = x_{5,6,7,8}$ we get Ineq.(68)

$$-1 + x^2 < 3 - 2r^2 - 9x^2 + 8r^2x^2 - 6r^2x^4 < 1 - x^2 \tag{68}$$

which leads us to following system of inequalities:

$$\begin{cases} 1 - r^2 - 4x^2 + 4r^2x^2 - 3r^2x^4 & < 0, \\ 2 - r^2 - 5x^2 + 4r^2x^2 - 3r^2x^4 & > 0. \end{cases} \tag{60}$$

For

$$x_{5,6} = \pm\sqrt{\frac{r + \sqrt{r^2 - 4}}{2r}} \tag{61}$$

we have

$$x^2 = \frac{r + \sqrt{r^2 - 4}}{2r}. \tag{62}$$

Substitute Eq.(62) into first inequality of system - Ineq.(60):

$$1 - r^2 - 2 \cdot \frac{r + \sqrt{r^2 - 4}}{r} + 2r(r + \sqrt{r^2 - 4}) - \frac{3}{2}(r^2 + r\sqrt{r^2 - 4} - 2) < 0, \tag{63}$$

$$2 - \frac{1}{2}r^2 - 2\frac{\sqrt{r^2 - 4}}{r} + \frac{1}{2}r\sqrt{r^2 - 4} < 0, \tag{64}$$

$$\frac{r(4 - r^2) - (4 - r^2)\sqrt{r^2 - 4}}{r} < 0, \tag{65}$$

$$\frac{(4 - r^2)(r - \sqrt{r^2 - 4})}{r} < 0. \tag{66}$$

If r>2 then $r–\sqrt{r^2 - 4} > 0$ and $4–r^2 < 0$, hence $r \in (2,+\infty)$.
If r<–2 then $r–\sqrt{r^2 - 4} < 0$ and $4–r^2 < 0$, hence $r \in (-\infty,–2)$.
Substitute Eq.(62) into second inequality of system Ineq.(60):

$$2 - r^2 - 5 \cdot \frac{r + \sqrt{r^2 - 4}}{2r} + 2r(r + \sqrt{r^2 - 4}) - \frac{3}{2}(r^2 + r\sqrt{r^2 - 4} - 2) > 0, \tag{67}$$

$$5 - r^2 - \frac{5\sqrt{r^2 - 4}}{r} + r\sqrt{r^2 - 4} > 0, \tag{68}$$

$$\frac{(5 - r^2) \cdot (r - \sqrt{r^2 - 4})}{r} > 0. \tag{69}$$

If r>2 then $r–\sqrt{r^2 - 4} > 0$. From Ineq.(69) follows that $5–r^2 > 0$. This takes place if $r \in (2,\sqrt{5})$.
If r<–2 then $r–\sqrt{r^2 - 4} < 0$. From Ineq.(69) follows that $5–r^2 > 0$. This takes place if $r \in (–\sqrt{5},–2)$.
So we got that the points

$$x_{5,6} = \pm\sqrt{\frac{r + \sqrt{r^2 - 4}}{2r}} \tag{70}$$

are periodic attractors if $r \in (–\sqrt{5},–2) \cup (2,\sqrt{5})$.
Let's go back to the system of inequalities of Ineq.(60). For

$$x_{7,8} = \pm\sqrt{\frac{r - \sqrt{r^2 - 4}}{2r}} \tag{71}$$

we have

$$x^2 = \frac{r - \sqrt{r^2 - 4}}{2r}. \tag{72}$$

Substitute Eq.(72) it into first inequality Ineq.(60):

$$1 - r^2 - 2 \cdot \frac{r - \sqrt{r^2 - 4}}{r} + 2r(r - \sqrt{r^2 - 4}) - \frac{3}{2}(r^2 - r\sqrt{r^2 - 4} - 2) < 0, \tag{73}$$

$$2 - \frac{1}{2}r^2 + 2\frac{\sqrt{r^2 - 4}}{r} - \frac{1}{2}r\sqrt{r^2 - 4} < 0, \tag{74}$$

$$\frac{r(4 - r^2) + (4 - r^2)\sqrt{r^2 - 4}}{r} < 0, \tag{75}$$

$$\frac{(4 - r^2)(r + \sqrt{r^2 - 4})}{r} < 0. \tag{76}$$

If r>2 then $r+\sqrt{r^2 - 4} > 0$ and $4–r^2 < 0$, hence $r \in (2,+\infty)$.
If r<–2 then $r+\sqrt{r^2 - 4} < 0$ and $4–r^2 < 0$, hence $r \in (-\infty,–2)$.
Substitute Eq.(72) it into second inequality of Ineq.(60):

$$2 - r^2 - 5\frac{r - \sqrt{r^2 - 4}}{2r} + 2r(r - \sqrt{r^2 - 4}) - \frac{3}{2}(r^2 - r\sqrt{r^2 - 4} - 2) > 0, \tag{77}$$

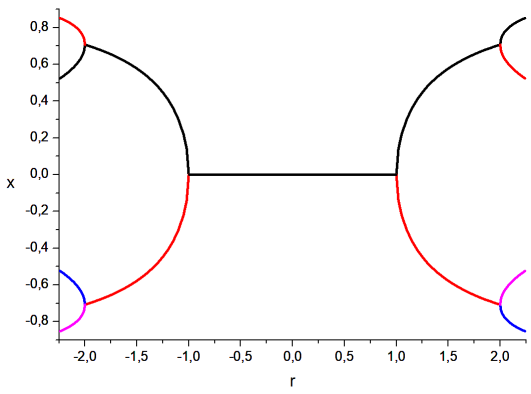


Fig. 4. Bifurcation diagram for recurrent sequence $x_{t+1} = r\cdot\, x_t\, \cdot(1 - x_t^2)$ at the interval $(-\sqrt{5},\sqrt{5})$.

$$5 - r^2 + \frac{5\sqrt{r^2-4}}{r} - r\sqrt{r^2-4} > 0, \tag{78}$$

$$\frac{(5-r^2)\cdot(r+\sqrt{r^2-4})}{r} > 0. \tag{79}$$

If r>2 then $r+\sqrt{r^2-4} > 0$. From Ineq.(69) follows that $5–r^2 > 0$. This takes place if $r\in(2,\sqrt{5})$. If r<–2 then $r+\sqrt{r^2-4} < 0$. From Ineq.(69) follows that $5–r^2 > 0$. This takes place if $r\in(-\sqrt{5},–2)$. So we got that the points

$$x_{7,8} = \pm\sqrt{\frac{r-\sqrt{r^2-4}}{2r}} \tag{80}$$

are periodic attractors if $r\in(-\sqrt{5},–2)\cup(2,\sqrt{5})$.
Thus, we have established that points

$$x_{5,6,7,8} = \pm\sqrt{\frac{r\pm\sqrt{r^2-4}}{2r}} \tag{81}$$

are periodic attractors if $r\in(-\sqrt{5},-2)\cup(2,\sqrt{5})$.
For recurrent sequence $x_{t+1} = r\cdot x_t\cdot(1 - x_t^2)$ the bifurcation diagram at the interval $(-\sqrt{5},\sqrt{5})$ was depicted as presented in Fig. 4. Bifurcation diagram with period doubling 1, 2, 4 was obtained in analytical form solving 8-th degree polynomial equation. Strong expressed period doubling is presented at points r=1 and r=2.
Figs. 5, 6, 7, 8 represent graphical solutions of two functions $y = x$ and $y = \Phi^2(x)$ at different r: r=2.0, r=2.2, r=2.23, and r=2.3 respectively. For r=2.0 (Fig. 5), five crossing points are present, but zero periodic attractors. For r=2.2 (Fig. 6), nine crossing points are present, but four of them are periodic attractors. For r=2.23 (Fig. 7), nine crossing points are present, but four of them are periodic attractors. For r=2.3 (Fig. 8), nine crossing points are present, but none of them are periodic attractors.

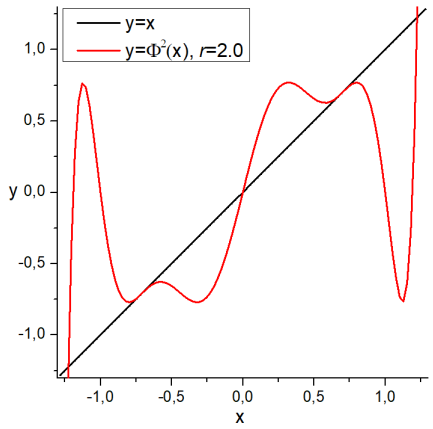


Fig. 5. $y = x$ and $y = \Phi^2(x)$ at r=2. Five crossing points, but zero periodic attractors.

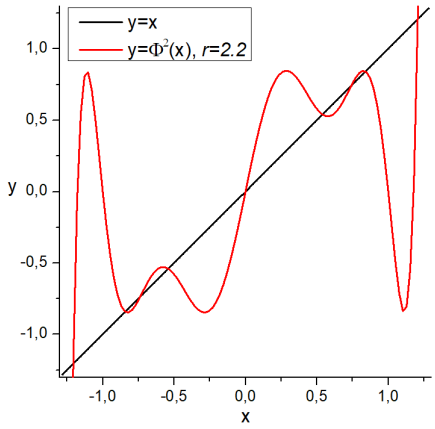


Fig. 6. $y = x$ and $y = \Phi^2(x)$ at r=2.2. Nine crossing points, four of them are periodic attractors.

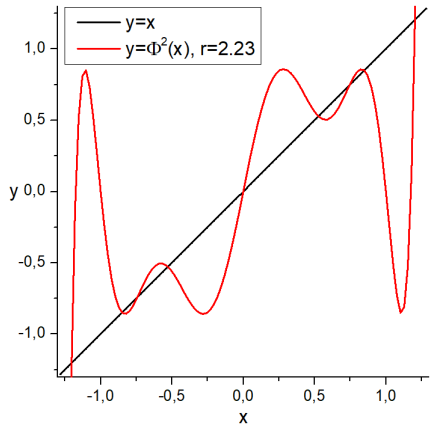


Fig. 7. $y = x$ and $y = \Phi^2(x)$ at r=2.23. Nine crossing points, four of them are periodic attractors.

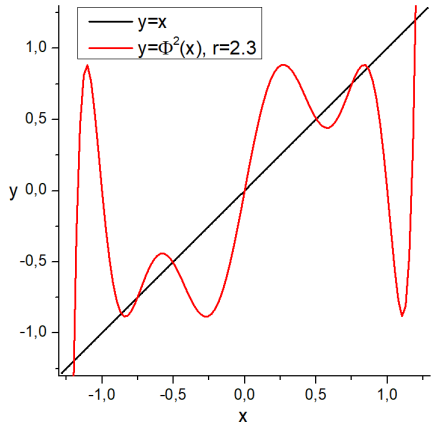


Fig. 8. $y = x$ and $y = \Phi^2(x)$ at r=2.3. Nine crossing points, but none of them are periodic attractors.

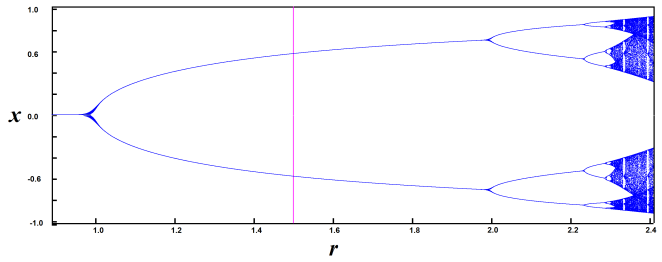


Fig. 9. Finite State Diagram of sequence $x_{t+1} = r \cdot x_t \cdot (1 - x_t^2)$, t=0, 1, 2, ... Values in interval (0;1) (top branch) were generated using x_0=0.3, and values in interval (-1;0) (bottom branch) were generated using x_0=–0.3.

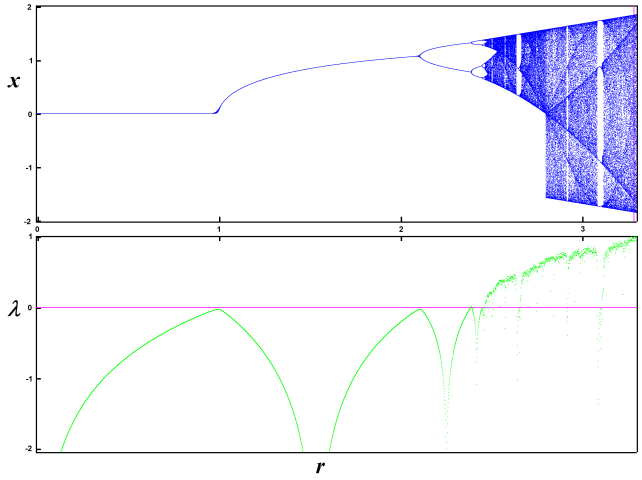


Fig. 10. Finite State Diagram (top) and distribution of Lyapunov exponent index (bottom) of recurrent sequence $x_{t+1} = r{\cdot}x_t \cdot\cos x_t$, t=0, 1, 2, . . . , x_0 = 0.5.

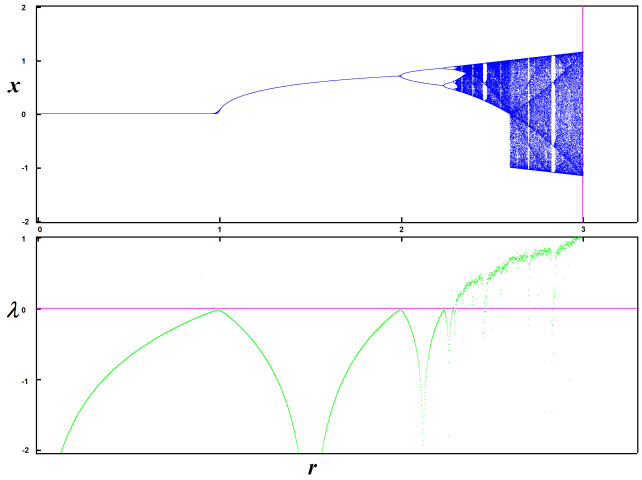


Fig. 11. Finite State Diagram (top) and distribution of Lyapunov exponent index (bottom) of recurrent sequence $x_{t+1} = r{\cdot}x_t{\cdot}(1 - x_t^2)$, t=0, 1, 2, . . . , x_0 = 0.5.

5. Approximation methods for visualization

Initial estimation of recurrent sequence properties could be done using previously presented tool *QUATTRO* [27] where several behaviours such as Finite State Diagram and of Lyapunov exponent index are presented as the fingerprints of recurrent relations. Fig. 9 represent the Finite State Diagram generated using recurrent sequence $x_{t+1} = r \cdot x_t \cdot (1 - x_t^2)$. Values in interval (0;1) (top branch) were generated using initial value x_0=0.3, and corresponding values in interval (–1;0) (bottom branch) were generated using x_0=–0.3.

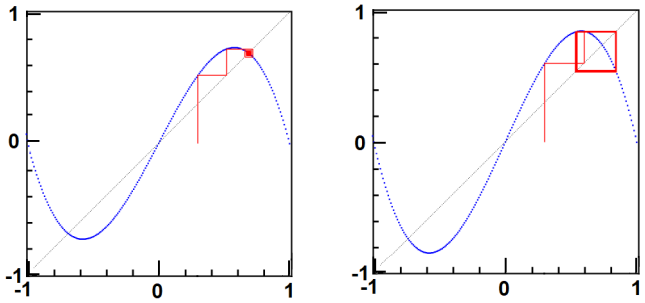
Figs. 10 and 11 represent the Finite State Diagram and distribution of Lyapunov exponent index $\lambda(x_0)$ of two recurrent relations $x_{t+1} = r \cdot x_t \cdot \cos x_t$, and $x_{t+1} = r \cdot x_t \cdot (1 - x_t^2)$, t=0, 1, 2, . . ., respectively. In both cases, initial value x_0 = 0.5. Both Finite State Diagrams look quite similar. Their different ranges of values are explained by the normalization of the scales, see Eq.(20).

Fig. 11 (bottom) represents the distribution of Lyapunov Exponent index:

$$\lambda(x_0) = \lim_{t\to\infty}\left(\frac{1}{t}\sum_{i=0}^{t-1}\ln\left|\Phi'(x_i)\right|\right). \qquad (82)$$

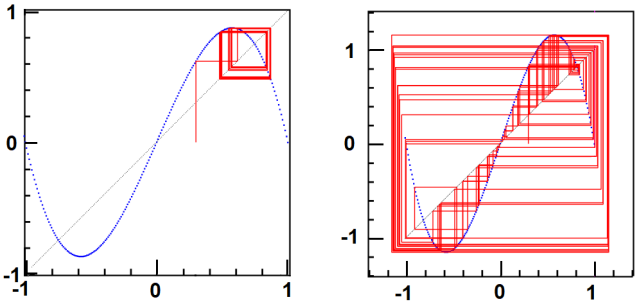
At r=1.5, value of distribution $\lambda(x_0)$ tends to $-\infty$ which indicates the absence of chaotic dependence on initial x_0. Distribution of Lyapunov Exponent index $\lambda(x_0)$ characterizes the behaviour of chaotic dynamics as well as various forms of stabilization or synchronization. Positive value of Lyapunov exponent index indicates chaotic behaviour of the sequence according to sensitive dependence on initial x_0.

Figs. 12, 13 represent CoWeb plots of the recurrent sequence $x_{t+1} = r \cdot x_t \cdot(1 - x_t^2)$, t=0, 1, 2, . . ., x_0 = 0.5, when r_1=1.9, r_2=2.2, r_3=2.26, r_4=3.0. Different character of sequence convergence could be established. 50 iterations is enougth to describe the trending manner. Fig. 12a represents visualization of the fixed point attractor (r_1=1.9). Figs. 12b and 13a represent visualization of two types of periodic attractors (period doubling 2 for r_2=2.2 and period doubling 4 for r_3=2.26). For r_3=3.0, chaotic dependence occurs - see Fig. 13b.



a) r_1=1.9; b) r_2=2.2.

Fig. 12. CoWeb plot of recurrent sequence $x_{t+1} = r \cdot x_t \cdot (1 - x_t^2)$, t=0, 1, 2, . . ., x_0 = 0.3. 50 iterations were used.



a) r_3=2.26; b) r_4=3.0.

Fig. 13. CoWeb plot of recurrent sequence $x_{t+1} = r \cdot x_t \cdot (1 - x_t^2)$, t=0, 1, 2, . . ., x_0 = 0.3. 50 iterations were used.

Conclusions

By analysing of recurrent sequence generated by trigonometric function $F(x) = r \cdot x \cdot \cos x$ some problems arise related to second composite $F^2(x)$ due to the complexity of expressions. Using MacLaurin approximation for function $\cos x$, the dynamic behavior of the recurrent sequence was studied in the same manner as discrete analogue of Verhulst equation [10]. We presented the bifurcation diagram for recurrent sequence generated by power function $\Phi(x) = r \cdot x \cdot (1 - x^2)$. Function $\Phi(x)$ and its second composite $\Phi^2(x)$ were used. Analytical solutions of equation $x = \Phi^2(x)$ allows us to establish the fixed point attractors and periodic attractors in the interval $(-\sqrt{5}, \sqrt{5})$. Bifurcation diagram with period doubling 1, 2, 4 was obtained in analytical form solving 8-th degree polynomial equation and compared with Finite State diagram as an aproximate analogue.

Authors' contributions

Jelena Kozmina formulated general idea, derived mathematical formulas and validated results. Alytis Gruodis presented literature review, realized the graphical implementation and prepared analysis of data. Both authors reviewed and approved the final manuscript.

Conflicts of interest

Authors declared at they have no conflicts of interest.

References

1. Bruno Gonpe Tafo, J.; Nana, L.; Tabi, C. B.; Kofané, T. C. (2020) Nonlinear Dynamical Regimes and Control of Turbulence through the Complex Ginzburg-Landau Equation - *Research Advances in Chaos Theory* IntechOpen - doi:10.5772/intechopen.88053.
2. Tzamal- Odysseas, M. (2014) Energy transfer and dissipation in nonlinear oscillators. PhD theses - Aristotle University of Thessaloniki, Greece, 2014.
3. Elaydi, S. (2005) An introduction to difference equations 3rd ed. - Springer Science: Business Media, Inc., 2005.
4. Devaney , R. L. (2020) A First Course in Chaotic Dynamical Systems. Theory and Experiment. 2nd Edition - Taylor & Francis Group, LLC, 2020.
5. Chen, Y.; Qian, Y.; Cui, X. (2022) Time series reconstructing using calibrated reservoir computing - *Scientific Reports* 12 (2022) 16318 - https://doi.org/10.1038/s41598-022-20331-3
6. Tronci, S.; Giona, M.; Baratti, R. (2003) Reconstruction of chaotic time series by neural models: a case study - *Neurocomputing* 55 (2003)581-591 - https://doi.org/10.1016/S0925-2312(03)00394-1.
7. Boutsinas, B.; Vrahatis, M.N. (2001) Artificial nonmonotonic neural networks - *Artificial Intelligence* 132(2001)1-38 - https://doi.org/10.1016/S0004-3702(01)00126-6.
8. Bakas, I.; Kontoleon, K.J. (2021) Performance Evaluation of Artificial Neural Networks (ANN) Predicting Heat Transfer through Masonry Walls Exposed to Fire - *Applied Science* 11 (2021)11435 - https://doi.org/10.3390/app112311435.
9. Gómez-Ramos, E.; Venegas-Martínez, F. (2013) A Review of Artificial Neural Networks: How Well Do They Perform in Forecasting Time Series? - *Analitika, Revista de analisis estadistico* 3 (2013)7-15
10. Kozmina, Y. (2018) Discrete Analogue of the Verhulst Equation and Attractors. Methodological Aspects of Teaching - *Innovative Infotechnologies for Science, Business and Education* 1(24) (2018) 3-12.
11. Kozmina, Y.; Gruodis, A. (2020) QUATTRO-20: advanced tool for estimation of the recurrent sequences - In: 18th International Conference "Information Tehnologies and Management", April 23-24, 2020, ISMA University of Applied Science, Riga, Latvia.
12. Kozmina, J.; Gruodis, A. (2023) Tool QUATTRO-20 for Examining of the Recurrent Sequencies Generated by Discrete Analogue of the Verhulst Equation. - *Applied Business: Issues & Solutions* 1(2023)16–29 - https://doi.org/10.57005/ab.2023.1.3.
13. Dosly, O.; Pechancova, S. (2006) Trigonometric Recurrence Relations and Tridiagonal Trigonometric Matrices - *International Journal of Difference Equations* 1 (2006) 19–29.
14. Brooke, K.; Saucedo, D.; Xu, C. (2017) Second-Order Linear Recurrence Relations and Periodicity - *The Onyx Review: The Interdisciplinary Research Journal* 2(2017) 7-12.
15. Farris, M.; Luntzlara, N.; Miller, S.; Lily, S.; Mengxi, W. (2021) Recurrence relations and Benford's law - *Statistical Methods & Applications* 797 (2021) 1613-981X - https://doi.org/10.1007/s10260-020-00547-1
16. Būdienė, G.; Gruodis, A. (2016) Zipf and related scaling laws. 3. Literature overview of multidisciplinary applications (from informational aspects to energetic aspects) - *Innovative Infotechnologies for Science, Business and Education* ISSN 2029-1035 – 2(21)(2016)12-19.
17. Andrianov, I.; Starushenko, G.; Kvitka, S.; Khajiyeva, K. (2019) The Verhulst-Like Equations: Integrable OΔE and ODE with Chaotic Behavior - *Symmetry* 11 (2019) 1446 - doi:10.3390/sym11121446.
18. Gutierrez, M. R.; Reyes, M.A.; Rosu, H.C. (2010) A note on Verhulst's logistic equation and related logistic maps- *J. Phys. A* 43 (2010) 205204.
19. Ragulskis, M.; Navickas, Z. (2011) The rank of a sequence as an indicator of chaos in discrete nonlinear dynamical systems - *Communications in Nonlinear Science and Numerical Simulation* 16(2011) 2894-2906 - https://doi.org/10.1016/j.cnsns.2010.10.008.
20. Hikihara, T.; Holmes, P.; Kambe, T.; Rega, G. (2012) Introduction to the focus issue: Fifty years of chaos: Applied and theoretical - *Chaos* 22(2012) 047501 - https://doi.org/10.1063/1.4769035
21. Ditto, W.; Munakata, T. (1995) Principles and Applications of Chaotic Systems - *Communications of the ACM* 38(1995) 96-102.
22. Nosrati, K.; Volos, C. (2018) Bifurcation Analysis and Chaotic Behaviors of Fractional-Order Singular Biological Systems - In: Nonlinear Dynamical Systems with Self-Excited and Hidden Attractors. Eds. Viet-Thanh Pham, Sundarapandian Vaidyanathan, Christos Volos, Tomasz Kapitaniak - Series: Studies in Systems, Decision and Control, Volume 133 - Springer International Publishing AG 2018.
23. Toker, D.; Sommer; F. T.; D'Esposito, M. (2020) A simple method for detecting chaos in nature - *Communications Biology* (2020) 3:11 - https://doi.org/10.1038/s42003-019-0715.
24. Fehrle, D.; Heiberger, C.; Huber, J. (2020) Polynomial chaos expansion: Efficient evaluation and estimation of computational models - *BGPE* Discussion Paper, No. 202, Friedrich-Alexander-Universität Erlangen-Nürnberg, Erlangen und Nürnberg - http://hdl.handle.net/10419/237993
25. Kozmina, Y.; Gruodis, A. (2019) Number generation based on the chaotic sequences - In: The 17th International Scientific Conference "Information Technologies and Management - 2019", April 25-26, 2019, ISMA, Riga, Latvia - *Nano Technologies and Computer Modelling* (2019)17-18.
26. Alawida, M.; Samsudin, A.; Teh, J. S. (2019) Enhancing Unimodal Digital Chaotic Maps through Hybridization - *Nonlinear Dynamics* 96 (2019) 601–613 - https://doi.org/10.1007/s11071-019-04809-w
27. Kozmina, J.; Gruodis A. (2020) *QUATTRO-20* - WinApi program. - https://github.com/Alytis/QUATTRO-20.

This page intentionally left blank...

This page intentionally left blank...

This page intentionally left blank...